Well-Ordering Principles in Proof Theory and Reverse Mathematics

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- One needs a general theory of *ordinal representations systems*.

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   2.3 β-models and comprehension

The "Big" Five

For many mathematical theorems τ, there is a weakest natural subsystem $S(\tau)$ of $\mathbf{Z_2}$ such that $S(\tau)$ proves τ.
Moreover, it has turned out that $S(\tau)$ often belongs to a small list of specific subsystems of $\mathbf{Z_2}$. Reverse Mathematics has singled out five subsystems of $\mathbf{Z_2}$:

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- $(\Pi^1_1\mathbf{-CA})_0$ Π^1_1-Comprehension

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where f is a standard proof theoretic function from ordinals to ordinals.

There are by now several examples of functions f where the statement **WOP**(f) has turned out to be equivalent to one of the theories of reverse mathematics over a weak base theory (usually $\mathbf{RCA}_0$).

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Abstract property WO of real object $2^{\mathfrak{X}}$ versus existence of abstract sets **ACA**.

Cantor's Representation of Ordinals

Theorem (Cantor, 1897) For every ordinal $\beta > 0$ there exist unique ordinals $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ such that

$$\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_n}. \tag{1}$$

The representation of β in (1) is called the **Cantor normal form**.

We shall write $\beta =_{CNF} \omega^{\beta_1} + \cdots \omega^{\beta_n}$ to convey that $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_k$.

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- $\beta < \varepsilon_0$ has a Cantor normal form with exponents $\beta_i < \beta$ and these exponents have Cantor normal forms with yet again smaller exponents. As this process must terminate, ordinals $< \varepsilon_0$ can be coded by natural numbers.

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- Hindman's Theorem and the Auslander/Ellis theorem are provable in $\mathbf{ACA}_0^+$.

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- B. Afshari, R.: *Reverse Mathematics and Well-ordering Principles: A pilot study*, APAL 160 (2009) 231-237.

The ordering $<_{\varepsilon_{\mathfrak{X}}}$

Let $\mathfrak{X} = \langle X, <_X \rangle$ be an ordering where $X \subseteq \mathbb{N}$.
$<_{\varepsilon_{\mathfrak{X}}}$ and its field $|\varepsilon_{\mathfrak{X}}|$ are inductively defined as follows:

1. $0 \in |\varepsilon_{\mathfrak{X}}|$.
2. $\varepsilon_u \in |\varepsilon_{\mathfrak{X}}|$ for every $u \in X$, where $\varepsilon_u := \langle 0, u \rangle$.
3. If $\alpha_1, \ldots, \alpha_n \in |\varepsilon_{\mathfrak{X}}|$, $n > 1$ and $\alpha_n \leq_{\varepsilon_{\mathfrak{X}}} \ldots \leq_{\varepsilon_{\mathfrak{X}}} \alpha_1$, then

$$\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$$

   where $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} := \langle 1, \langle \alpha_1, \ldots, \alpha_n \rangle \rangle$.
4. If $\alpha \in |\varepsilon_{\mathfrak{X}}|$ and α is not of the form ε_u, then $\omega^\alpha \in |\varepsilon_{\mathfrak{X}}|$, where $\omega^\alpha := \langle 2, \alpha \rangle$.

1. $0 <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$ for all $u \in X$.
2. $0 <_{\varepsilon_{\mathfrak{X}}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ for all $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$.
3. $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \varepsilon_v$ if $u, v \in X$ and $u <_x v$.
4. If $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$, $u \in X$ and $\alpha_1 <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$ then $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} <_{\varepsilon_{\mathfrak{X}}} \varepsilon_u$.
5. If $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \in |\varepsilon_{\mathfrak{X}}|$, $u \in X$, and $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \alpha_1$ or $\varepsilon_u = \alpha_1$, then $\varepsilon_u <_{\varepsilon_{\mathfrak{X}}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$.
6. If $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ and $\omega^{\beta_1} + \ldots + \omega^{\beta_m} \in |\varepsilon_{\mathfrak{X}}|$, then

$$\begin{aligned}
&\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} <_{\varepsilon_{\mathfrak{X}}} \omega^{\beta_1} + \ldots + \omega^{\beta_m} \text{ iff} \\
&n < m \wedge \forall i \leq n \; \alpha_i = \beta_i \quad \text{or} \\
&\exists i \leq \min(n, m)[\alpha_i <_{\varepsilon_{\mathfrak{X}}} \beta_i \wedge \forall j < i \;\; \alpha_j = \beta_j].
\end{aligned}$$

Let $\varepsilon_{\mathfrak{X}} = \langle |\varepsilon_{\mathfrak{X}}|, <_{\varepsilon_{\mathfrak{X}}} \rangle$.

Ordinal representation systems

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Hardy gives explicit representations for all ordinals $< \omega^2$.

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- He applied two new operations to **continuous increasing functions** on ordinals:
  - **Derivation**
  - **Transfinite Iteration**
- Let ON be the class of ordinals. A (class) function $f : \mathrm{ON} \to \mathrm{ON}$ is said to be **increasing** if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$ and **continuous** (in the order topology on ON) if

$$f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

holds for every limit ordinal λ and increasing sequence $(\alpha_\xi)_{\xi<\lambda}$.

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- The **derivative** f' of a function $f : \mathrm{ON} \to \mathrm{ON}$ is the function which enumerates in increasing order the solutions of the equation

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also called the **fixed points** of f.
- If f is a normal function,

$$\{\alpha : f(\alpha) = \alpha\}$$

is a proper class and f' will be a normal function, too.

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$$f_\lambda(\xi) = \xi^{th} \text{ element of } \bigcap_{\alpha<\lambda} \{\text{Fixed points of } f_\alpha\} \qquad \text{for } \lambda \text{ limit.}$$

The Feferman-Schütte Ordinal Γ_0

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  is the famous ordinal Γ_0 which Feferman and Schütte determined to be the least ordinal 'unreachable' by certain autonomous progressions of theories.

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(i) $\varphi_{\alpha_1}(\beta_1) = \varphi_{\alpha_2}(\beta_2)$ holds iff one of the following conditions is satisfied:

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(ii) $\varphi_{\alpha_1}(\beta_1) < \varphi_{\alpha_2}(\beta_2)$ holds iff one of the following conditions is satisfied:

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- Friedman's proof uses computability theory and also some proof theory. Among other things it uses a result which states that if $P \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that

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- Friedman's proof uses computability theory and also some proof theory. Among other things it uses a result which states that if $P \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that
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- R. and Weiermann, *Reverse mathematics and well-ordering principles*, Computability in Context: Computation and Logic in the Real World (S. B. Cooper and A. Sorbi, eds.) (Imperial College Press, 2011) 351–370.

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- **Definition.** $\mathfrak{M}$ is a countable coded ω-model of T if

$$\mathfrak{X} = \{(C)_n \mid n \in \mathbb{N}\}$$

for some $C \subseteq \mathbb{N}$ where $(C)_n = \{k \mid 2^n 3^k \in C\}$.

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R., ω-models and well-ordering principles. In: N. Tennant (ed.): *Foundational Adventures: Essays in Honor of Harvey M. Friedman.* (2014)

Lemma

$\mathbf{ATR}_0$ can be axiomatized via a single sentence Π^1_2 sentence

$$\forall X\, C(X)$$

where $C(X)$ is Σ^1_1.

Proof: This is a standard result. See Simpson's book.

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(iii) Henceforth a **sequent** will be a finite set of L_2-formulas *without free number variables*.

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(iv) A sequent is **reducible** or a **redex** if it is not axiomatic and contains a formula which is not a literal.

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(iv) Every reducible Γ_i with $i < k$ is of the form

$$\Gamma_i', E, \Gamma_i''$$

where E is not a literal and Γ_i' contains only literals.
E is said to be the **redex** of Γ_i.

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Aiming at a contradiction, suppose that $\mathcal{D}_Q$ is a well-founded tree. Let $\mathfrak{X}_0$ be the Kleene-Brouwer ordering on $\mathcal{D}_Q$. Then $\mathfrak{X}_0$ is a well-ordering. In a nutshell, the idea is that a well-founded $\mathcal{D}_Q$ gives rise to a derivation of the empty sequent (contradiction) in the infinitary proof systems $\mathcal{T}_Q^\infty$ from R.: *The strength of Martin-Löf type theory with a superuniverse. Part II.* Archive for Mathematical Logic 40 (2001) 207-233.

The Big Veblen Number

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- Veblen extended this idea first to arbitrary **finite numbers of arguments**, but then also to **transfinite numbers of arguments**, with the proviso that in, for example

$$\Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta),$$

only a finite number of the arguments

$$\alpha_\nu$$

may be non-zero.

- Veblen singled out the ordinal $E(0)$, where $E(0)$ is the least ordinal $\delta > 0$ which cannot be named in terms of functions

$$\Phi_\ell(\alpha_0, \alpha_1, \ldots, \alpha_\eta)$$

with $\eta < \delta$, and each $\alpha_\gamma < \delta$.

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- Define a set of ordinals $\mathfrak{B}$ closed under successor such that with each limit $\lambda \in \mathfrak{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathfrak{B}$ of length $\tau_\lambda \leq \mathfrak{B}$ and $\lim_{\xi < \tau_\lambda} \lambda[\xi] = \lambda$.

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- Let Ω be the first uncountable ordinal. A hierarchy of functions $(\overset{\mathfrak{B}}{\varphi}_\alpha)_{\alpha \in \mathfrak{B}}$ is then obtained as follows:

$$\overset{\mathfrak{B}}{\varphi}_0(\beta) = 1 + \beta \qquad \overset{\mathfrak{B}}{\varphi}_{\alpha+1} = \left(\overset{\mathfrak{B}}{\varphi}_\alpha\right)'$$

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The Bachmann-Howard ordinal is the order-type of $\mathrm{OT}(\vartheta) \cap \Omega$.

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2. *Every set is contained in a countable coded ω-model of* **BI**.

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An ω-model $\mathfrak{A}$ is a β-model if the concept of well ordering is absolute with respect to $\mathfrak{A}$.

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The question arises whether the methodology can be extended to more complex axiom systems, in particular to those characterizable via β-models?

First of all, to get equivalences one has to climb up in the type structure. Given a functor

$$F : (\mathbb{LO} \to \mathbb{LO}) \to (\mathbb{LO} \to \mathbb{LO}),$$

where $\mathbb{LO}$ is the class of linear orderings, we consider the statement:

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There is also a variant of $\mathbf{WOPP}(F)$ which should basically encapsulate the same "power". Given a functor

$$G : (\mathbb{LO} \to \mathbb{LO}) \to \mathbb{LO}$$

consider the statement:

$$\mathbf{WOPP}_1(G) : \qquad \forall f \in (\mathbb{LO} \to \mathbb{LO})\ [\mathbf{WOP}(f) \to \mathbf{WOP}(G(f))].$$