Reverse Mathematics of some Principles related to Partial Orders

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August 13$^{\text{th}}$ 2019

Reverse Mathematics

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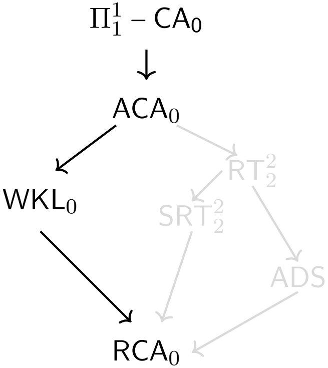

Π^1_1 comprehension

Arithmetic comprehension

Stable Ramsey's Theorem for pairs

"Every infinite linear order contains an infinite ascending chain or an infinite descending chain"

Recursive (Δ^0_1) comprehension

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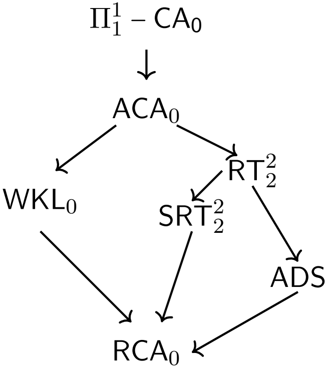

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Definitions and statement

Recall that, given a poset $(P, <_P)$:

- a *chain* $C \subset P$ is a linearly ordered subset of P.
- an *antichain* $A \subset P$ is a set such that for every $a, b \in A$, a and b are incomparable (so $a \nless_P b$ and $b \nless_P a$).
- the *width* of a poset P is the supremum of the cardinalities of the antichains of P.

Theorem (Rival and Sands, 1980)

(RS-po) *Let P be an infinite partial order of finite width. Then there exists an infinite chain $C \subset P$ such that for every $p \in P$, p is comparable with 0 or infinitely many elements of C.*

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Rival-Sands for graphs

One might wonder where such a statement comes from. The principle RS-po was introduced as a refinement of a result about graphs:

Theorem (Rival and Sands, 1980)

(RS-g) *Let G be an infinite graph, then there exists an infinite subgraph $H \subset G$ such that every vertex $g \in G$ is adjacent to 0, 1 or infinitely many vertices of H.*
Moreover, every $h \in H$ is adjacent to 0 or infinitely many other elements of H.

This result is interesting because it is, in some sense, a modification of Ramsey's Theorem.

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From graphs to posets

As Rival and Sands pointed out, the result above takes a much nicer form under the assumption that G is the comparability graph of a poset P of finite width.

With this setting in mind, we could rephrase RS-po as follows:

Theorem (Rival and Sands, 1980)

If G_P is the comparability graph of an infinite poset P of finite width, then there exists a complete subgraph $H \subset G_P$ such that every $p \in P$ is adjacent to 0 or infinitely many elements of H.

The theorem above is not, to the best of our knowledge, a trivial corollary of RS-g.

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Remarks on the proof of RS-po in ZFC

The original proof of the theorem given by Rival and Sands actually gives a stronger result:

Theorem (Rival and Sands, 1980)

(sRS-po) *If P is an infinite poset of finite width, then there is a chain C of order type ω or ω^* such that every element $p \in P$ is comparable with 0 or infinitely many (and hence cofinitely many) elements of C.*

A direct translation of the original proof requires $\Pi^1_1 - CA_0$ to be carried out (although by a standard result of Reverse Mathematics it cannot be that sRS-po and $\Pi^1_1 - CA_0$ are equivalent over RCA_0). The study of the strength of this principle is work in progress.

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An easier proof of RS-po

It is possible to give an easier proof (by contradiction) of RS-po. The key ideas of this proof are the following:

- We observe that if P contains a copy of $\mathbb{Z}$, then that copy is a solution.
- We decompose P into k chains (where k is the width of P). Performing this step already requires WKL_0 (see Hirst, 1987).
- Then, we separate every chain into its well-founded and reverse well-founded part. This requires ACA_0.
- Finally, by "counterexample chasing", one sees that it is impossible to prevent every ω or ω^*-chain from being a solution. Again, this step requires ACA_0.

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Principles related to RS-po

In order to better understand RS-po, it is useful to analyse some simpler principles related to it.

Definition

- For every $k \in \mathbb{N}$, we define RS-po$_k$ as the restriction of RS-po to posets of width at most k.
- For every $k \in \mathbb{N}$, we define RS-po$_k^{CD}$ the statement "If P is a poset that can be decomposed into k chains, then the conclusion of RS-po holds".

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Recursive chain decomposition

Theorem (Kierstead, 1981)

(RCA_0) *For every $k \in \mathbb{N}$, if P is a poset of width k, then it can be decomposed into at most 5^k chains.*

The original proof by Kierstead made essential use of strong induction principles. The proof was massaged into an argument in RCA_0 thanks to the help of Keita Yokoyama.

It follows that $\mathsf{RCA}_0 \vdash \forall k \in \mathbb{N}(\mathsf{RS\text{-}po}^{\mathsf{CD}}_{5^k} \to \mathsf{RS\text{-}po}_k)$. Moreover, thanks to the result above, in order to study the strength of RS-po, it is enough to analyse the simpler principle $\forall k \mathsf{RS\text{-}po}^{\mathsf{CD}}_k$.

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Results for standard k's

The main result concerning the strength of $\mathsf{RS\text{-}po}_k$ is the following:

Theorem (Fiori Carones, Marcone, Shafer, S.)

For every $k \in \omega, k \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{ADS} \leftrightarrow \mathsf{RS\text{-}po}_k \leftrightarrow \mathsf{RS\text{-}po}_k^{\mathsf{CD}}$.

The reversal above is actually a proof that $\mathsf{RCA}_0 \vdash \mathsf{RS\text{-}po}_3^{\mathsf{CD}} \rightarrow \mathsf{ADS}$. To the best of our knowledge, this seems to be the first case of a genuine mathematical statement being equivalent to ADS.

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The case $k = 2$

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Theorem (Fiori Carones, Marcone, Shafer, S.)

$\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{RS\text{-}po}^{\mathsf{CD}}_2 \to \mathsf{SADS}$

Using Dilworth's Theorem, we have

Corollary

$\mathsf{WKL}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{RS\text{-}po}_2$

By the main result of Chong, Yang, and Slaman, 2010, it follows that $\mathsf{RS\text{-}po}_2$ does not imply ADS, and so is strictly weaker than $\mathsf{RS\text{-}po}_3$.

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References

Chong, C. T., Y. Yang, and T. Slaman (2010). "The metamathematics of stable Ramsey's theorem for pairs". In: URL: https://math.berkeley.edu/~slaman/papers/SRT22.pdf.

Hirst, Jeffry L. (1987). "Combinatorics in Subsystems of Second order Arithmetic". PhD thesis. The Pennsylvania State University.

Kierstead, Henry A. (1981). "An Effective Version of Dilworth's Theorem". In: *Transactions of the American Mathematical Society* 268.1, pp. 63–77. ISSN: 00029947. URL: http://www.jstor.org/stable/1998337.

Rival, Ivan and Bill Sands (1980). "On the Adjacency of Vertices to the Vertices of an Infinite Subgraph". In: *Journal of the London Mathematical Society* s2-21.3, 393–400. DOI: 10.1112/jlms/s2-21.3.393.