Descending Sequences of Hyperdegrees and the Second Incompleteness Theorem

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Main theme of the talk

There is a connection between Gödel's second incompleteness theorem and well-foundedness of certain computability-theoretic partial orders

Some Definitions

Hyperarithmetic reducibility

$X \leq_H Y$ means X is Δ^1_1 definable using Y as a parameter

Analogous to Turing reducibility (Δ^1_1 instead of Δ^0_1)

Hyperjump of X

$\mathcal{O}^X$ means the Π^1_1-complete set relative to X

Analogous to the Turing jump (Π^1_1 instead of Σ^0_1)

Church-Kleene ordinal

ω_1^X means the least ordinal with no presentation computable from X

A Theorem

Theorem

There is no sequence of reals $A_0, A_1, \ldots$ such that for each n

$$\mathcal{O}^{A_{n+1}} \leq_H A_n.$$

First proof

By results of Spector, if $\mathcal{O}^A \leq_H B$ then $\omega_1^A < \omega_1^B$. So if $A_0, A_1, A_2, \ldots$ was such a sequence then we would have

$$\omega_1^{A_0} > \omega_1^{A_1} > \omega_1^{A_2} > \ldots$$

We can replace the use of ordinals in the previous proof with an appeal to the second incompleteness theorem. This lowers the complexity of the proof (in the sense of reverse math).

Some Suggestive Facts

Definition

A β-model is an ω-model of second order arithmetic that is correct about all Σ^1_1 facts

Fact

(ACA_0 proves) $\mathcal{O}^X$ exists $\implies X$ is contained in a β-model

Fact

(ACA_0 proves) β-models satisfy ACA_0

Incompleteness $\implies$ Well-foundedness

An alternative proof?

- Work in the theory $T = \mathrm{ACA}_0 +$ "there is such a sequence"
- Let $A_0, A_1, \ldots$ be such a descending sequence
- $\mathcal{O}^{A_1}$ exists so there is a β-model containing A_1
- The β-model satisfies ACA_0 and contains $A_1, A_2, \ldots$
- The tail of a descending sequence is still a descending sequence and being a descending sequence is absolute between β-models
- So the β-model satisfies T
- So T proves its own consistency
- By the second incompleteness theorem, T is inconsistent

Incompleteness $\implies$ Well-foundedness

An alternative proof?

- Work in the theory $T = \mathrm{ACA}_0 +$ "there is such a sequence"
- Let $A_0, A_1, \ldots$ be such a descending sequence
- $\mathcal{O}^{A_1}$ exists so there is a β-model containing A_1
- The β-model satisfies ACA_0 and contains $A_1, A_2, \ldots$ but not necessarily a real coding this sequence
- The tail of a descending sequence is still a descending sequence and being a descending sequence is absolute between β-models
- So the β-model satisfies T
- So T proves its own consistency
- By the second incompleteness theorem, T is inconsistent

Actually, this proof is not quite correct

Incompleteness $\Longrightarrow$ Well-foundedness

A Correct Proof

The main idea is to show that if there is a descending sequence then there is one which is relatively simple—e.g. hyperarithmetic in A_1. To do this, use the Kleene basis theorem.

Kleene Basis Theorem

(ACA_0 proves) If X is a real such that $\mathcal{O}^X$ exists and if φ is a Σ^1_1 formula and there is some real Y such that $\varphi(X, Y)$ holds then there is some real Y such that $Y \leq_T \mathcal{O}^X$

So it suffices to show that there is a nonempty $\Sigma^1_1(A_2)$ class consisting only of descending sequences

Incompleteness $\implies$ Well-foundedness

A Correct Proof

The main idea is to show that if there is a descending sequence then there is one which is relatively simple—e.g. hyperarithmetic in A_1. To do this, use the Kleene basis theorem.

It suffices to show that there is a nonempty $\Sigma^1_1(A_2)$ class consisting only of descending sequences

One way to do this is to pick a countable coded β-model, M, that is hyperarithmetic in A_2 and use the formula that says

"X is a sequence of reals so that for each n, X_n is in M and $M \models (\mathcal{O}^{X_{n+1}}$ exists and $\mathcal{O}^{X_{n+1}} \leq_H X_n)$"

(In fact this formula is actually arithmetic in M, though that doesn't change the proof.)

Incompleteness $\Longrightarrow$ Well-foundedness

Observation

This proof shows that the theorem is provable in ACA_0 which is not apparent from the first proof.

Incompleteness $\implies$ Well-foundedness

General Strategy

- Work in some theory T and assume there is a descending sequence
- Show that there is a model of T containing a tail of the sequence
- Tail of a descending sequence is still a descending sequence
- Conclude that $T +$ "there is a descending sequence" proves its own consistency

Main Difficulty

Need to pick a theory T that is weak enough that the existence of a descending sequence guarantees models of the theory exist but strong enough to prove that the descending sequence guarantees this.

Incompleteness $\implies$ Well-foundedness

Example 1: Hyperdegrees

There is no sequence of hyperdegrees, each hyp above the hyperjump of the next

Example 2: Turing degrees

Under certain conditions there is no sequence of Turing degrees, each Turing above the Turing jump of the next

Theorem (Steel)

If P is an arithmetic relation then there is no sequence $A_0, A_1, A_2, \ldots$ such that for each n

- A_{n+1} is the unique X such that $P(A_n, X)$
- $A'_{n+1} \leq_T A_n$

Steel's original proof used only recursion theory, but Harvey Friedman later gave a proof along the lines of the general strategy outlined here

Incompleteness $\implies$ Well-foundedness

We have seen how the second incompleteness theorem can be used to prove the well-foundedness of some computability-theoretic partial-orders

Actually, the connection goes both ways.

Well-foundedness $\implies$ Incompleteness

The well-foundedness of computability-theoretic partial orders can sometimes imply semantic versions of the second incompleteness theorem

Semantic Versions of Second Incompleteness

Gödel's second incompleteness theorem

A consistent theory cannot prove its own consistency

Consistent = has a model

Semantic version of second incompleteness

If T has a model then it has a model with no models coded in it.

Typically T is a theory of second order arithmetic which is strong enough to prove Gödel's completeness theorem.

By changing what type of models we consider, we can get statements that do not trivially follow from the usual second incompleteness theorem.

Well-foundedness $\implies$ Incompleteness

Theorem (Mummert–Simpson)

If T is an arithmetically axiomatized theory in the language of second order arithmetic such that T has a β-model, then T has a β-model that contains no countable coded β-models of T

The original proof by Mummert and Simpson uses the regular second incompleteness theorem.

We can replace the appeal to the second incompleteness theorem by using the ordinals—in particular the well-foundedness of the partial order from the first half of the talk. This also yields a slightly stronger result where T does not have to be arithmetically axiomatizable.

Well-foundedness $\implies$ Incompleteness

Theorem (Mummert–Simpson)

If T is a theory in the language of second order arithmetic such that T has a β-model, then T has a β-model that contains no countable coded β-models of T

Proof sketch

- Suppose not. Find a sequence of β-models $M_0, M_1, M_2, \ldots$ such that each M_{n+1} is coded in M_n
- Since M_n is a β-model it is correct about all Π^1_1 facts about M_{n+1}
- So $\mathcal{O}^{M_{n+1}}$ is arithmetic in M_n
- So $\mathcal{O}^{M_{n+1}} \leq_H M_n$, contradicting the first theorem in this talk

Well-foundedness $\implies$ Incompleteness

Observation

This is slightly stronger than the theorem originally proved by Mummert and Simpson, which was only for arithmetically axiomatized theories.

Well-foundedness $\implies$ Incompleteness

Main Observation

The lack of a minimal model can imply a descending sequence in some computability-theoretic partial order.

Example 1: β-models

If T is any theory such that T has a β-model then T has a β-model with no countable coded β-models of T.

Example 2: ω-models of a theory extending ACA_0 (Steel)

If T is an arithmetically axiomatized theory extending ACA_0 such that T has an ω-model then T has an ω-model with no countable coded ω-models of T.

Conclusion

In some cases, well-foundedness can be replaced with the second incompleteness theorem and vice-versa. Switching from one to the other can result in a sharpened version of the original theorem.

Thank you!