ON RANKS FOR FAMILIES OF THEORIES OF FINITE ABELIAN GROUPS

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Topologies, closures, generating sets e-spectra, and ranks for families of theories: references

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Topologies, closures, generating sets e-spectra, and ranks for families of theories: references

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Approximating families of theories

Throughout we consider families $\mathcal{T}$ of complete first-order theories of a language $\Sigma = \Sigma(\mathcal{T})$. For a sentence φ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$ being the φ*-neighbourhood* in $\mathcal{T}$.

Definition. Let $\mathcal{T}$ be a family of theories and T be a theory, $T \notin \mathcal{T}$. The theory T is called $\mathcal{T}$*-approximated*, or *approximated by* $\mathcal{T}$, or $\mathcal{T}$*-approximable*, or a *pseudo-*$\mathcal{T}$*-theory*, if for any formula $\varphi \in T$ there is $T' \in \mathcal{T}$ such that $\varphi \in T'$.

If T is $\mathcal{T}$-approximated then $\mathcal{T}$ is called an *approximating family* for T, theories $T' \in \mathcal{T}$ are *approximations* for T, and T is an *accumulation point* for $\mathcal{T}$.

e-minimal families of theories

An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

It was shown in [S.V. Sudoplatov, Approximations of theories, arXiv:1901.08961v1 [math.LO], 2019.] that any e-minimal family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Ranks for families of theories

For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and for infinite families $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.
For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.
If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.
We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.
If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

Totally transcendental families of theories[1]

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.
If $\mathcal{T}$ is totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

[1] S.V. Sudoplatov, Ranks for families of theories and their spectra, arXiv:1901.08464v1 [math.LO], 2019.

Totally transcendental families of theories[2]

Theorem 1

For any family $\mathcal{T}$, $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}(\mathrm{Cl}_E(\mathcal{T}))$, and if $\mathcal{T}$ is nonempty and e-totally transcendental then $\mathrm{ds}(\mathcal{T}) = \mathrm{ds}(\mathrm{Cl}_E(\mathcal{T}))$.

Theorem 2

For any family $\mathcal{T}$ with $|\Sigma(\mathcal{T})| \leq \omega$ the following conditions are equivalent:
(1) $|\mathrm{Cl}_E(\mathcal{T})| = 2^\omega$;
(2) $e\text{-}\mathrm{Sp}(\mathcal{T}) = 2^\omega$;
(3) $\mathrm{RS}(\mathcal{T}) = \infty$.

[2] S.V. Sudoplatov, Ranks for families of theories and their spectra, arXiv:1901.08464v1 [math.LO], 2019.

Definable subfamilies[3]

Definition. Let $\mathcal{T}$ be a family of first-order complete theories in a language Σ. For a set Φ of Σ-sentences we put $\mathcal{T}_\Phi = \{T \in \mathcal{T} \mid \Phi \subseteq T\}$. A family of the form $\mathcal{T}_\Phi$ is called *d-definable* (in $\mathcal{T}$). If Φ is a singleton $\{\varphi\}$ then $\mathcal{T}_\varphi = \mathcal{T}_\Phi$ is called *s-definable*.

Theorem 3

Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, $\alpha \in \{0, 1\}$, $n \in \omega \setminus \{0\}$. Then there is a d-definable subfamily $\mathcal{T}_\Phi$ such that $\mathrm{RS}(\mathcal{T}_\Phi) = \alpha$ and $\mathrm{ds}(\mathcal{T}_\Phi) = n$.

[3] N.D. Markhabatov, S.V. Sudoplatov, Definable subfamilies of theories, related calculi and ranks, arXiv:1901.08961v1 [math.LO], 2019.

Definable subfamilies[4]

Recall that a subfamily $\mathcal{T}_0$ of $\mathcal{T}$ is called d_∞-*definable* if $\mathcal{T}_0$ is a union, possibly infinite, of d-definable subfamilies of $\mathcal{T}$.

Theorem 4

Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable subfamily $\mathcal{T}^* \subset \mathcal{T}$ such that $\mathrm{RS}(\mathcal{T}^*) = \alpha$ and $\mathrm{ds}(\mathcal{T}^*) = n$.

[4]N.D. Markhabatov, S.V. Sudoplatov, Definable subfamilies of theories, related calculi and ranks, arXiv:1901.08961v1 [math.LO], 2019.

Theories of abelian groups

Let $\mathcal{A}$ be an abelian group in the language $\Sigma = \langle +^{(2)}, -^{(1)}, 0^{(0)} \rangle$. Then $k\mathcal{A}$ denotes its subgroup $\{ka \mid a \in A\}$ and $\mathcal{A}[k]$ denotes the subgroup $\{a \in A \mid ka = 0\}$. Let P be the set of all prime numbers. If $p \in P$ and $pA = \{0\}$ then $\mathrm{dim}\mathcal{A}$ denotes the dimension of the group $\mathcal{A}$, considered as a vector space over a field with p elements. The following numbers, for arbitrary $p \in P$ and $n \in \omega \setminus \{0\}$ are called the *Szmielew invariants* for the group A:[5] [6]

[5]Ershov Yu.L., Palyutin E.A. Mathematical logic. — Moscow : FIZMATLIT, 2011.

[6]Szmielew W. Elementary properties of Abelian groups // Fund. Math., 1955, Vol. 41, P. 203–271.

Theories of abelian groups

$$\alpha_{p,n}(\mathcal{A}) = \min\{\dim((p^n\mathcal{A})[p]/(p^{n+1}\mathcal{A})[p]), \omega\},$$

$$\beta_p(\mathcal{A}) = \min\{\inf\{\dim((p^n\mathcal{A})[p] \mid n \in \omega\}, \omega\},$$

$$\gamma_p(\mathcal{A}) = \min\{\inf\{\dim((\mathcal{A}/\mathcal{A}[p^n])/p(\mathcal{A}/\mathcal{A}[p^n])) \mid n \in \omega\}, \omega\},$$

$\varepsilon(\mathcal{A}) \in \{0, 1\}$, and $\varepsilon(\mathcal{A}) = 0 \Leftrightarrow (n\mathcal{A} = \{0\}$ for some $n \in \omega, n \neq 0)$.

We denote the set of Szmielew invariants by **Szm**.

Theories of abelian groups

It is known that two abelian groups are elementary equivalent if and only if they have same Szmielew invariants. Besides, the following proposition holds.[7]

Proposition 1

Let for any p and n the cardinals $\alpha_{p,n}$, β_p, $\gamma_p \leq \omega$, and $\varepsilon \in \{0, 1\}$ be given. Then there is an abelian group $\mathcal{A}$ such that the Szmielew invariants $\alpha_{p,n}(\mathcal{A})$, $\beta_p(\mathcal{A})$, $\gamma_p(\mathcal{A})$, and $\varepsilon(\mathcal{A})$ are equal to $\alpha_{p,n}$, β_p, γ_p, and ε, respectively, if and only if the following conditions hold:
(1) if for prime p the set $\{n \mid \alpha_{p,n} \neq 0\}$ is infinite then $\beta_p = \gamma_p = \omega$;
(2) if $\varepsilon = 0$ then for any prime p, $\beta_p = \gamma_p = 0$ and the set $\{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$ is finite.

[7] Ershov Yu.L., Palyutin E.A. Mathematical logic. — Moscow : FIZMATLIT, 2011.

Theories of abelian groups

We denote by $\mathbb{Q}$ the additive group of rational numbers, $\mathbb{Z}_{p^n}$ — the cyclic group of the order p^n, $\mathbb{Z}_{p^\infty}$ — the quasi-cyclic group of all complex roots of 1 of degrees p^n for all $n \geq 1$, R_p — the group of irreducible fractions with denominators which are mutually prime with p. The groups $\mathbb{Q}$, $\mathbb{Z}_{p^n}$, R_p, $\mathbb{Z}_{p^\infty}$ are called *basic*. Below the notations of these groups will be identified with their universes. Since abelian groups with same Szmielew invariants have same theories, any abelian group $\mathcal{A}$ is elementary equivalent to a group

$$\oplus_{p,n} \mathbb{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p \mathbb{Z}_{p^\infty}^{(\beta_p)} \oplus \oplus_p R_p^{(\gamma_p)} \oplus \mathbb{Q}^{(\varepsilon)}, \qquad (1)$$

where $\mathcal{B}^{(k)}$ denotes the direct sum of k subgroups isomorphic to a group $\mathcal{B}$. Thus, any theory of an abelian group has a model being a direct sum of based groups. The groups of form (1) are called *standard*.

Theories of abelian groups

Recall that any complete theory of an abelian group is based by the set of positive primitive formulas, reduced to the set of the following formulas:

$$\exists y(m_1x_1 + \ldots + m_nx_n \approx p^k y), \quad (2)$$

$$m_1x_1 + \ldots + m_nx_n \approx 0, \quad (3)$$

where $m_i \in \mathbb{Z}$, $k \in \omega$, p is a prime number. Formulas (2) and (3) allow to witness that Szmielew invariants defines theories of abelian groups modulo Proposition 1.

In view of Proposition 1 and equations (2) and (3) we have the following:

Remark. Theories of abelian groups are forced by sentences implied by formulas of form (2) and (3) and describing dimensions with respect to $\alpha_{p,n}$, β_p, γ_p, ε as well as bounds for orders p^k of elements and possibilities for divisions of elements by p^k. Moreover, distinct values of Szmielew invariants are separated by some sentences modulo Proposition 1. Hence, counting ranks of families of theories of abelian groups it suffices to consider sentences separating Szmielew invariants.

Families of theories of finite abelian groups

Consider the family $\mathcal{T}_{A,\mathrm{fin}}$ of all theories of finite abelian groups. Clearly, $\mathcal{T}_{A,\mathrm{fin}}$ is countable corresponding to tuples of non-zero values of $\alpha_{p,n}$. By Proposition 1 the E-closure of $\mathcal{T}_{A,\mathrm{fin}}$ produces theories, of infinite abelian groups, with some $\alpha_{p,n}$ and $\beta_p = \gamma_p = \omega$. Since $\beta_p = \gamma_p = \omega$ can be obtained independently with respect to distinct p, we have $|\mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}})| = 2^\omega$. Applying Theorem 2 we have:

Theorem 5

$\mathrm{RS}(\mathcal{T}_{A,\mathrm{fin}}) = \infty$.

Pseudofinite theories

Recall that an infinite structure $\mathcal{M}$ is *pseudofinite*[8] if every sentence true in $\mathcal{M}$ has a finite model. Here the theory $\mathrm{Th}(\mathcal{M})$ is also called *pseudofinite*.

Now we consider Szmielew invariants of theories in $\mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}})$. Since theories of finite groups can not generate new theories of finite groups and finite abelian groups have finitely many nonzero values $\alpha_{p,n}$, and $\beta_p = \gamma_p = \varepsilon = 0$ for any prime p, it suffices to study theories of pseudofinite groups, i.e., theories in $\mathcal{T}_{A,\mathrm{pf}} = \mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}}) \setminus \mathcal{T}_{A,\mathrm{fin}}$.

[8] Rosen E. Some Aspects of Model Theory and Finite Structures // The Bulletin of Symbolic Logic. 2002. Vol. 8, No. 3. P. 380–403.
Macpherson D. Model theory of finite and pseudofinite groups // Archive for Mathematical Logic. 2018. Vol. 57, No. 1–2. P. 159–184.

Theorem 6

For any theory T of abelian groups the following conditions are equivalent:
(1) $T \in \mathcal{T}_{A,\mathrm{pf}}$;
(2) T has some infinite $\alpha_{p,n}$, or some $\beta_p = \gamma_p = \omega$, or $\varepsilon = 1$, moreover, for all nonzero values β_p and γ_p, $\beta_p = \gamma_p = \omega$;
(3) T has infinite models, and all nonzero values β_p and γ_p imply $\beta_p = \gamma_p = \omega$.

Pseudofinite theories of abelian groups

Notice that by Theorem 6 infinite standard groups

$$\oplus_{p,n}\mathbb{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p \mathbb{Z}_{p^\infty}^{(\omega)} \oplus \oplus_p R_p^{(\omega)} \oplus \mathbb{Q}^{(\varepsilon)}$$

and, in particular, the group $\mathbb{Q}$ are pseudofinite.

Theorem 6 immediately implies:

Corollary

If a theory T of an abelian group has a positive natural value β_p or γ_p then models of T are not pseudofinite.

Since $\mathrm{Th}(\mathbb{Z})$ has values $\gamma_p = 1$, the group $\mathbb{Z}$ is not pseudofinite.

We again consider the family $\mathcal{T}_{A,\mathrm{fin}}$ of all theories of finite abelian groups and the family $\mathcal{T}_{A,\mathrm{pf}} = \mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}}) \setminus \mathcal{T}_{A,\mathrm{fin}}$ of pseudofinite abelian groups.

Theorem 7

Let α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a subfamily $\mathcal{T} \subset \mathcal{T}_{A,\mathrm{fin}}$ such that $\mathrm{RS}(\mathcal{T}) = \alpha$, $\mathrm{ds}(\mathcal{T}) = n$, and $\mathrm{Cl}_E(\mathcal{T}) \subset \mathcal{T}_{A,\mathrm{fin}} \cup \mathcal{T}_{A,\mathrm{pf}}$ is d-definable with $(\mathrm{RS}(\mathrm{Cl}_E(\mathcal{T})), \mathrm{ds}(\mathrm{Cl}_E(\mathcal{T})) = (\alpha, n)$.