ON COMPOSITION OF STRUCTURES AND COMPOSITIONS OF THEORIES

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Aims

- A generalization of classical notions of graph composition and semigroup composition till arbitrary structures of predicate languages, as well as till (multi)algebras $\mathfrak{P}_{\nu(R)}$ of binary isolating formulas of given theories over families R of types;
- Study the basedness and model-theoretic properties of compositions of structures and their elementary theories;
- Representation of algebras for binary isolating formulas as a composition both in general case and for linear preorders, circular preorders, series of finite structures.

Definition. Let T be a complete theory, $\mathcal{M} \models T$. Consider types $p(x), q(y) \in S(\varnothing)$, realized in $\mathcal{M}$, and all (p, q)-*semi-isolating*, or (p, q)-*preserving* formulas $\varphi(x, y)$ of T, i. e., formulas for which there is $a \in M$ such that $\models p(a)$ and $\varphi(a, y) \vdash q(y)$. Now, for each such a formula $\varphi(x, y)$, we define a binary relation $R_{p,\varphi,q} \rightleftharpoons \{(a, b) \mid \mathcal{M} \models p(a) \wedge \varphi(a, b)\}$. If $(a, b) \in R_{p,\varphi,q}$, then (a, b) is called a (p, φ, q)-*arc*. If $\varphi(a, y)$ is principal (over a), the (p, φ, q)-arc (a, b) is also *principal*. If besides $\varphi(x, b)$ is a principal formula (over b) then the set $[a, b] \rightleftharpoons \{(a, b), (b, a)\}$ is a *principal* (p, φ, q)-edge.

Definitions

For types $p(x), q(y) \in S(\varnothing)$, we denote by $\mathrm{PF}(p, q)$ the set

$$\{\varphi(x, y) \mid \varphi(a, y) \text{ is a principal formula, } \varphi(a, y) \vdash q(y), \text{ where } \models p(a)\}.$$

Let $\mathrm{PE}(p, q)$ be the set of all pairs $(\varphi(x, y), \psi(x, y))$ of formulas in $\mathrm{PF}(p, q)$ such that for any (some) realization a of p the sets of solutions for $\varphi(a, y)$ and $\psi(a, y)$ coincide.
Clearly, $\mathrm{PE}(p, q)$ is an equivalence relation on the set $\mathrm{PF}(p, q)$.

Definitions

Thus the quotient $\mathrm{PF}(p,q)/\mathrm{PE}(p,q)$ is represented as a disjoint union of sets $\mathrm{PFS}(p,q)$ and $\mathrm{PFN}(p,q)$, where $\mathrm{PFS}(p,q)$ consists of $\mathrm{PE}(p,q)$-classes corresponding to principal edges and $\mathrm{PFN}(p,q)$ consists of $\mathrm{PE}(p,q)$-classes corresponding to irreversible principal arcs.

Definitions

Let T be a complete theory, $U = U^- \dot\cup \{0\} \dot\cup U^+$ be an alphabet of cardinality $\geq |S(T)|$, consisting of *negative elements* $u^- \in U^-$, *positive elements* $u^+ \in U^+$, and zero 0. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$. The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$. Elements of U are called *labels*.

Definitions

Let $\nu(p,q)\colon \mathrm{PF}(p,q)/\mathrm{PE}(p,q) \to U$ be an injective *labelling functions*, $p(x)$, $q(y) \in S(\varnothing)$, for which negative elements correspond to classes in $\mathrm{PFN}(p,q)/\mathrm{PE}(p,q)$ and non-negative elements correspond to classes in $\mathrm{PFS}(p,q)/\mathrm{PE}(p,q)$ such that 0 is defined only for $p = q$ and is represented by the formula $(x \approx y)$, $\nu(p) \rightleftharpoons \nu(p,p)$. We additionally suppose that $\rho_{\nu(p)} \cap \rho_{\nu(q)} = \{0\}$ for $p \neq q$ (where, as usual, we denote by ρ_f the image of the function f) and $\rho_{\nu(p,q)} \cap \rho_{\nu(p',q')} = \varnothing$ if $p \neq q$ and $(p,q) \neq (p',q')$. Labelling functions with the properties above as well families of these functions are said to be *regular*. Below we shall consider only regular labelling functions and their regular families.

Definitions

We denote by $\theta_{p,u,q}(x,y)$ a formula in $\mathrm{PF}(p,q)$ with the label $u\in\rho_{\nu(p,q)}$. If a type p is fixed and $p=q$ then a formula $\theta_{p,u,q}(x,y)$ is denoted by $\theta_u(x,y)$.
For types $p_1,p_2,\ldots,p_{k+1}\in S^1(\varnothing)$ and sets $X_1,X_2,\ldots,X_k\subseteq U$ of labels we denote by

$$P(p_1,X_1,p_2,X_2,\ldots,p_k,X_k,p_{k+1})$$

the set of all labels $u\in U$ corresponding to formulas $\theta_{p_1,u,p_{k+1}}(x,y)$ satisfying, for realizations a of p_1 and some $u_1\in X_1,\ldots,u_k\in X_k$, the following condition:

$$\theta_{p_1,u,p_{k+1}}(a,y)\vdash\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(a,y),$$

where

$$\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(x,y)\rightleftharpoons$$

$$\rightleftharpoons\exists x_2,x_3,\ldots x_k(\theta_{p_1,u_1,p_2}(x,x_2)\wedge\theta_{p_2,u_2,p_3}(x_2,x_3)\wedge\ldots$$

$$\ldots\wedge\theta_{p_{k-1},u_{k-1},p_k}(x_{k-1},x_k)\wedge\theta_{p_k,u_k,p_{k+1}}(x_k,y)).$$

Definitions

Thus the Boolean $\mathcal{P}(U)$ of U is the universe of an *algebra of distributions of binary isolating formulas* with k-ary operations

$$P(p_1, \cdot, p_2, \cdot, \ldots, p_k, \cdot, p_{k+1}),$$

where $p_1, \ldots, p_{k+1} \in S^1(\varnothing)$.

Definitions

If all types p_i equal to a type p then we write $P_p(X_1, X_2, \ldots, X_k)$ and $P_p(u_1, u_2, \ldots, u_k)$ as well as $\lfloor X_1, X_2, \ldots, X_k \rfloor_p$ and $\lfloor u_1, u_2, \ldots, u_k \rfloor_p$ instead of

$$P(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$$

and

$$P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$$

respectively.

We set $\mathfrak{P}_{\nu(p)} \rightleftharpoons \langle \mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}; \lfloor \cdot, \cdot \rfloor_p \rangle$. The groupoid $\mathfrak{P}_{\nu(p)}$ is called the *groupoid of binary isolating formulas over the labelling function* $\nu(p)$ or the $I_{\nu(p)}$*-groupoid*. Below the operation $\lfloor \cdot, \cdot \rfloor$ will be also denoted by $\cdot$ and we write uv instead of $u \cdot v$.

Definitions

Since by the choice of the label 0 for the formula $(x \approx y)$ the equalities $X \cdot \{0\} = X$ and $\{0\} \cdot X = X$ are true for any $X \subseteq \rho_{\nu(p)}$, the groupoid $\mathfrak{P}_{\nu(p)}$ has the unit $\{0\}$, and it is a monoid if the algebra is right semi-associative. We have

$$Y \cdot Z = \bigcup \{yz \mid y \in Y, z \in Z\}$$

for any sets $Y, Z \in \mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}$ in this structure.

Groupoids $\mathfrak{P}_{\nu(p)}$ are naturally extensible to groupoids $\mathfrak{P}_{\nu(R)}$ for binary isolating formulas on nonempty families $R \subseteq S^1(\emptyset)$ of types.

Definitions

A groupoid $\mathfrak{P} = \langle \mathcal{P}(U) \setminus \{\varnothing\}; \cdot \rangle$ is called an *I-groupoid* if it satisfies the following conditions:

• the set $\{0\}$ is the unit of the groupoid $\mathfrak{P}$;

• the operation $\cdot$ of the groupoid $\mathfrak{P}$ is generated by the function $\cdot$ on elements in U such that every elements $u, v \in U$ define a nonempty set $(u \cdot v) \subseteq U$: for any sets $X, Y \in \mathcal{P}(U) \setminus \{\varnothing\}$ the following equality holds:

$$X \cdot Y = \bigcup \{x \cdot y \mid x \in X, y \in Y\};$$

• if $u < 0$ then the sets $u \cdot v$ and $v \cdot u$ consist of negative elements for any $v \in U$;

• if $u > 0$ and $v > 0$ then the set $u \cdot v$ consists of non-negative elements;

Definitions

• for any $u > 0$ there is a unique *inverse* element $u^{-1} > 0$ such that $0 \in (u \cdot u^{-1}) \cap (u^{-1} \cdot u)$;

• if a positive element u belongs to a set $v_1 \cdot v_2$ then u^{-1} belongs to $v_2^{-1} \cdot v_1^{-1}$;

• for any elements $u_1, u_2, u_3 \in U$ the following inclusion holds:

$$(u_1 \cdot u_2) \cdot u_3 \supseteq u_1 \cdot (u_2 \cdot u_3),$$

and the strict inclusion

$$(u_1 \cdot u_2) \cdot u_3 \supset u_1 \cdot (u_2 \cdot u_3)$$

may be satisfied only for $u_1 < 0$ and $|u_2 \cdot u_3| \geq \omega$;

Definitions

• the groupoid $\mathfrak{P}$ contains the *deterministic* subgroupoid $\mathfrak{P}_d^{\geq 0}$ (being a monoid) with the universe $\mathcal{P}(U_d^{\geq 0}) \setminus \{\varnothing\}$, where

$$U_d^{\geq 0} = \{u \in U^{\geq 0} \mid u^{-1} \cdot u = \{0\}\};$$

any set $u \cdot v$ is a singleton for $u, v \in U_d^{\geq 0}$.

By the definition each I-groupoid $\mathfrak{P}$ contains I-subgroupoids $\mathfrak{P}^{\leq 0}$ and $\mathfrak{P}^{\geq 0}$ with the universes $\mathcal{P}(U^{-} \cup \{0\}) \setminus \{\varnothing\}$ and $\mathcal{P}(U^{+} \cup \{0\}) \setminus \{\varnothing\}$ respectively. The structure $\mathfrak{P}^{\geq 0}$ is a monoid.

Theorem (I.V. Shulepov, S.V. Sudoplatov, 2014)

For any I-groupoid $\mathfrak{P}$ there is a theory T with a type $p(x) \in S^1(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}$.

Recall[1] that the *composition* $\Gamma_1[\Gamma_2]$ *of graphs* $\Gamma_1 = \langle X_1; R_1\rangle$ and $\Gamma_2 = \langle X_2; R_2\rangle$ is the graph $\langle X_1 \times X_2; R\rangle$, where $((a_1, b_1), (a_2, b_2)) \in R$ if and only if some of the following conditions is met:
1) $(a_1, a_2) \in R_1$;
2) $a_1 = a_2$ and $(b_1, b_2) \in R_2$.

[1] F. Harary, Graph Theory. — Reading, Massachusetts : Addison-Wesley, 1969.

Definitions

Similarly, we consider the notion of monoid composition.
Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be monoids, for which 0 is the unit, $S_1 \cap S_2 = \{0\}$.
The *composition* or the *sequentially-annihilating band* $\mathcal{S}_1[\mathcal{S}_2]$ *of monoids* $\mathcal{S}_1$ and $\mathcal{S}_2$ is the algebra $\langle S_1 \cup S_2; \odot\rangle$, where $\langle S_1 \cup S_2; \odot\rangle \upharpoonright S_i = \mathcal{S}_i$ for $i = 1, 2$, and $u \odot v = v \odot u = u$ for $u \in S_1 \setminus \{0\}$ and $v \in S_2$.

Proposition (Lyapin E.S. Semigroups. — Providence : American Mathematical Society, 1974.)

Any sequentially-annihilating band $\mathcal{S}_1[\mathcal{S}_2]$ is a monoid.

Two theorems

Theorem (I.V. Shulepov, S.V. Sudoplatov, 2014)

For any group $\langle G; *\rangle$, where the universe consists of non-negative elements and 0 denotes the group unit, and for the monoid $\langle\{-1,0\}; +\rangle$ with the zero element 0 and the idempotent element -1, there is a theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that the monoid $\mathfrak{P}'_{\nu(p)}$ coincides with the monoid $\langle\{-1,0\}; +\rangle[\langle G; *\rangle]$.

Theorem (I.V. Shulepov, S.V. Sudoplatov, 2014)

For any group $\langle G; *\rangle$ consisting of non-negative elements with the unit element 0 and for the monoid $\langle\omega^*; +\rangle$ of non-positive integers, there exists a theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that the monoid $\mathfrak{P}'_{\nu(p)}$ coincides with the monoid $\langle\omega^*; +\rangle[\langle G; *\rangle]$.

Let $\mathcal{M}$ and $\mathcal{N}$ be structures of relational languages $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{N}}$ respectively. We define the *composition* $\mathcal{M}[\mathcal{N}]$ of $\mathcal{M}$ and $\mathcal{N}$ satisfying the following conditions:

1) $\Sigma_{\mathcal{M}[\mathcal{N}]} = \Sigma_{\mathcal{M}} \cup \Sigma_{\mathcal{N}}$;

2) $M[N] = M \times N$, where $M[N]$, M, N are universes of $\mathcal{M}[\mathcal{N}]$, $\mathcal{M}$, and $\mathcal{N}$ respectively;

3) if $R \in \Sigma_{\mathcal{M}} \setminus \Sigma_{\mathcal{N}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1, \ldots, a_n) \in R_{\mathcal{M}}$;

4) if $R \in \Sigma_{\mathcal{N}} \setminus \Sigma_{\mathcal{M}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$;

5) if $R \in \Sigma_{\mathcal{M}} \cap \Sigma_{\mathcal{N}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1, \ldots, a_n) \in R_{\mathcal{M}}$, or $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$.

Compositions of structures and compositions of theories

The theory $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is called the *composition* $T_1[T_2]$ of the theories $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$.
The notion of composition of structures is naturally spread for compositions $\mathfrak{P}_1[\mathfrak{P}_2]$ of algebras $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of binary isolating formulas.

By the definition, the composition $\mathcal{M}[\mathcal{N}]$ is obtained replacing each element of $\mathcal{M}$ by a copy of $\mathcal{N}$.

Proposition 1

If $\mathcal{M}$ and $\mathcal{N}$ have transitive automorphism groups then $\mathcal{M}[\mathcal{N}]$ has a transitive automorphism group, too.

e-definable compositions

Definition

The composition $\mathcal{M}[\mathcal{N}]$ is called *e-definable*, or equ-*definable*, if $\mathcal{M}[\mathcal{N}]$ has an $\emptyset$-definable equivalence relation E whose E-classes are universes of the copies of $\mathcal{N}$ forming $\mathcal{M}[\mathcal{N}]$. If the equivalence relation E is fixed, the *e*-definable composition is called E-definable.

e-definable compositions

Theorem 1

If a composition $\mathcal{M}[\mathcal{N}]$ is e-definable then:
1) $\mathcal{M}[\mathcal{N}] \simeq \mathcal{M}'[\mathcal{N}']$ if and only if $\mathcal{M} \simeq \mathcal{M}'$ and $\mathcal{N} \simeq \mathcal{N}'$;
2) $\mathcal{M}[\mathcal{N}] \equiv \mathcal{M}'[\mathcal{N}']$ if and only if $\mathcal{M} \equiv \mathcal{M}'$ and $\mathcal{N} \equiv \mathcal{N}'$.

Corollary

If a composition $\mathcal{M}[\mathcal{N}]$ is e-definable then the theory $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ uniquely defines the theories $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Th}(\mathcal{N})$, and the theories $\mathrm{Th}(\mathcal{M})$, $\mathrm{Th}(\mathcal{N})$ uniquely define the theory $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$.

Basedness of theories of E-definable compositions

Definition (E.A.Palyutin, U.Saffe, S.S.Starchenko)

A theory T is said to be Δ-*based*, where Δ is some set of formulas without parameters, if any formula of T is equivalent in T to a Boolean combination of formulas of Δ.

Proposition 2

If a composition $\mathcal{M}[\mathcal{N}]$ is E-definable then the theory $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is $(\Delta_1 \cup \Delta_2 \cup \widetilde{E})$-based, where $\mathrm{Th}(\mathcal{M})$ is Δ_1-based, $\mathrm{Th}(\mathcal{N})$ is Δ_2-based, and $\widetilde{E}$ is the set of formulas $E(x, y)$.

Countably categorical and minimal compositions

Theorem 2

A theory $T_1[T_2]$ of an infinite e-definable composition $\mathcal{M}[\mathcal{N}]$, where $T_1 = \mathrm{Th}(\mathcal{M})$, $T_2 = \mathrm{Th}(\mathcal{N})$, is $\aleph_0$-categorical if and only if one of $\mathcal{M}$ and $\mathcal{N}$ is finite and the other one is $\aleph_0$-categorical, or both structures are $\aleph_0$-categorical.

Theorem 3

An infinite e-definable composition $\mathcal{M}[\mathcal{N}]$ is (strongly) minimal if and only if $\mathcal{M}$ is a singleton and $\mathcal{N}$ is (strongly) minimal, or $\mathcal{M}$ is (strongly) minimal and $\mathcal{N}$ is finite.

Recall that a formula $\varphi(\bar{x}, \bar{y})$ of theory T is *stable* if there are no tuples $\bar{a}_n$, $\bar{b}_n$, $n \in \omega$, such that $\models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$. The theory T is called *stable* if every formula of T is stable.

Theorem 4

For any e-definable composition $\mathcal{M}[\mathcal{N}]$, the theory $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is stable if and only if $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Th}(\mathcal{N})$ are stable.

Algebras of binary formulas for *e*-definable compositions

Theorem 5

If a composition $\mathcal{M}[\mathcal{N}]$ is *e*-definable then the algebra $\mathfrak{P}_T$ of binary isolating formulas for $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is isomorphic to the composition $\mathfrak{P}_{T_1}[\mathfrak{P}_{T_2}]$ of the algebras $\mathfrak{P}_{T_1}$ and $\mathfrak{P}_{T_2}$ of binary isolating formulas for $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$.

Corollary

If a composition $\mathcal{M}[\mathcal{N}]$ is *e*-definable, $T_1 = \mathrm{Th}(\mathcal{M})$, $T_2 = \mathrm{Th}(\mathcal{N})$, and T_1, T_2 are transitive theories with algebras $\mathfrak{P}_{\nu(p)}$ and $\mathfrak{P}_{\nu'(p')}$, respectively, then the theory $T_1[T_2]$ has an algebra $\mathfrak{P}_{\nu''(p'')}$, with unique 1-type p'', which is isomorphic to $\mathfrak{P}_{\nu(p)}[\mathfrak{P}_{\nu'(p')}]$.

Algebras of binary formulas for linear preorders

Theorem 6

For any I-groupoid $\mathfrak{P}$, consisting of non-negative labels, there is a theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}_0[\mathfrak{P}]$, where $\mathfrak{P}_0$ is the algebra of binary formulas formulas for the theory of dense linear order without endpoints.

Theorem 7

For any I-groupoid $\mathfrak{P}$, consisting of non-negative labels, there is a theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \widehat{\mathfrak{P}}_0[\mathfrak{P}]$, where $\widehat{\mathfrak{P}}_0$ is the algebra of binary formulas formulas for nonprincipal 1-type of the Ehrenfeucht theory.

Theorem 8

For any I-groupoid $\mathfrak{P}$, consisting of non-negative labels, there is a theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}_{\mathbb{Z}}[\mathfrak{P}]$.

Circular preorders

Definition

A *circular* (or *cyclic*) order relation is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;

(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t [K(x, y, t) \vee K(t, y, z)])$;

(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

Weakly circularly minimal structure

Let $A \subseteq M$, where $\mathcal{M}$ is a circularly ordered structure. The set A is said to be *convex* if for any $a, b \in A$ the following holds: for any $c \in M$ with $K(a, c, b)$ we have $c \in A$ or for any $c \in M$ with $K(b, c, a)$ we have $c \in A$. $\mathcal{M}$ is said to be *weakly circularly minimal* if any definable (with parameters) subset of M is a finite union of convex sets.

Let $\mathcal{M}$ be an $\aleph_0$-categorical weakly circularly minimal structure, and $G := \mathrm{Aut}(\mathcal{M})$. Following standard group-theoretic terminology, we say G is *k-homogeneous*, where $k \in \omega$, if for any two k-element sets $A, B \subseteq M$ there is $g \in G$ with $g(A) = B$; also, G is called *highly homogeneous* if it is k-homogeneous for all $k \in \omega$. We say G is *k-transitive* if for distinct $a_1, a_2, \ldots, a_k \in M$ and distinct $b_1, b_2, \ldots, b_k \in M$ there is $g \in G$ with $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. By a *congruence* on $\mathcal{M}$ we mean a G-invariant equivalence relation on $\mathcal{M}$. We say G is *primitive* if it is 1-transitive and there are no non-trivial proper congruences on $\mathcal{M}$.

Theorem 9

For every natural $n \geq 1$ there exists an $\aleph_0$-categorical weakly circularly minimal structure with the primitive automorphism group so that the corresponding algebra of binary isolating formulas has exactly $n+1$ labels.

Theorem 10

The algebra $\mathfrak{P}_{\mathbb{Q}_n}$ of binary isolating formulas has the following rules of multiplication:
(1) For any label k with $0 \leq k \leq n$ we have $0 \cdot k = k \cdot 0 = \{k\}$;
(2) For any labels k_1, k_2 with $1 \leq k_1, k_2 \leq n$ we have
(2a) If $k_1 + k_2 \leq n$ then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_1 + k_2 - 1, k_1 + k_2\}$;
(2b) If $k_1 + k_2 - n = 1$ then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{0, 1, n\}$;
(2c) If $k_1 + k_2 - n = m$ for some $m \geq 2$ then
$k_1 \cdot k_2 = k_2 \cdot k_1 = \{m - 1, m\}$.

Algebras of binary formulas for circular preorders

Theorem 11

For any I-groupoid $\mathfrak{P}$, consisting of non-negative labels, there is a theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}_{\rm dco}[\mathfrak{P}]$, where $\mathfrak{P}_{\rm dco}$ is the algebra of binary isolating formulas for the dense circular order.

Theorem 12

For any I-groupoid $\mathfrak{P}$, consisting of non-negative labels, and for any natural $n \geq 2$, there is a theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}_{\mathbb{Z}_n}[\mathfrak{P}]$.

Examples of compositions of finite structures and compositions of finite algebras for binary formulas

Below we illustrate compositions of finite structures and compositions of finite algebras of binary formulas.

Let $\mathcal{M}_0$ be a two-element graph consisting of one edge. The theory $T_0 = \mathrm{Th}(\mathcal{M}_0)$ is transitive, and for the unique 1-type $p_0 \in S(T)$ and for its algebra $\mathfrak{L} = \mathfrak{P}_{\nu(p_0)}$ we have unique nonzero label, say 1, satisfying the following table:

$\cdot$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0\}$

Examples of compositions of finite structures and compositions of finite algebras for binary formulas

The composition $\mathcal{M}_0[\mathcal{M}_0]$ represents a 4-element complete graph K_4, it is not e-definable producing a transitive theory T_1 with a unique 1-type $p_1 \in S(T)$ and the algebra $\mathfrak{LL} = \mathfrak{P}_{\nu(p_1)}$ satisfying the following table:

$\cdot$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$

More generally, for any m-element complete graph K_m and n-element complete graph K_n of same language, $m, n \geq 2$, $K_m[K_n] \simeq K_{mn}$ and the theory $\mathrm{Th}(K_m[K_n])$ has the algebra $\mathfrak{LL}$ of binary isolating formulas.

Examples of compositions of finite structures and compositions of finite algebras for binary formulas

Now we modify the previous example considering graphs K_m and K_n such that edges $e_1 \in K_m$ and $e_2 \in K_n$ have distinct colors. For the algebra $\mathfrak{P}$ of binary isolating formulas for the theory $\mathrm{Th}(K_m[K_n])$ we obtain the following possibilities:
1) if $m = n = 2$ then $\mathfrak{P} = \mathfrak{L}[\mathfrak{L}']$, where $\mathfrak{L}' \simeq \mathfrak{L}$, has the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

2) if $m=2$ and $n>2$ then $\mathfrak{P}=\mathfrak{L}[\mathfrak{L}\mathfrak{L}]$ has the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

3) if $m > 2$ and $n = 2$ then $\mathfrak{P} = \mathfrak{L}\mathfrak{L}[\mathfrak{L}]$ has the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0, 1, 2\}$

4) if $m > 2$ and $n > 2$ then $\mathfrak{P} = \mathfrak{L}\mathfrak{L}[\mathfrak{L}\mathfrak{L}']$, where $\mathfrak{L}\mathfrak{L}' \simeq \mathfrak{L}\mathfrak{L}$, has the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

Examples of compositions of finite structures and compositions of finite algebras for binary formulas

Let C_m and C_n be undirected graphs forming cycles of lengths $m \geq 2$ and $n \geq 2$, respectively. Each $\mathrm{Th}(C_m)$ has the diameter $d_m = \left[\frac{m}{2}\right]$ and the algebra $\mathfrak{P}_{d_m}$ of binary isolating formulas with labels $0, 1, \ldots, d_m$ and the following rules for labels u and v:
$u \cdot v = |u \pm v| (\mathrm{mod}\, d_m)$, if m is even, or m is odd and $u + v \leq m$,

$$u \cdot v = \{(u + v - 1)(\mathrm{mod}\, d_m), |u - v|(\mathrm{mod}\, d_m)\},$$

if m is odd and $u + v > m$.

Examples of compositions of finite structures and compositions of finite algebras for binary formulas

Considering the graph $C_m[C_n]$ we have a non-e-definable combination of diameter d_m, and each copy of C_n has the diameter $\min\{d_n, 2\}$. For the theory $\mathrm{Th}(C_m[C_n])$ its algebra $\mathfrak{P}_{d_m,d_n}$ of binary isolating formulas has labels $0, 1, \ldots, d_m$ and the following rule: for any labels u, v, $u \cdot v$ consists of correspondent values for $\mathrm{Th}(C_m)$ as well as of u, if $v = 1$ or $v = 2$, and of v, if $u = 1$ or $u = 2$.

If C_m and C_n consist of edges of distinct colors then the combination $C_m[C_n]$ is e-definable, and the algebras $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of theories $T_1 = \mathrm{Th}(C_m)$ and $T_2 = \mathrm{Th}(C_n)$, respectively, produce the algebra $\mathfrak{P}_1[\mathfrak{P}_2]$ of the theory $T_1[T_2]$.