On connections between logic on words and limits of graphs

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Example (The space of types)

$$X_{\mathrm{FO}} \leftarrow\!\!\sim\!\sim\!\sim\!\sim\!\dashv \mathcal{LT}_{\mathrm{FO}}$$

Lindenbaum-Tarski algebra for $\mathrm{FO}(\sigma)$

- points are *types*
  $\approx$ equiv. classes of σ-structures M with $v \colon \mathrm{Var} \to M$
- basic opens $\widehat{\varphi} = \{\, [(M, v)] \mid M \models_v \varphi\}$, for $\varphi \in \mathrm{FO}(\sigma)$

Logic on Words

- Models: words $w \in A^*$ $\approx$ structures $(\{1, \ldots, |w|\}, <, P_a(x))_{a \in A}$

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Duality-theoretically [Gehrke, Petrișan, Reggio]:

$$\mathcal{B} \subseteq \mathcal{P}((A \times 2)^*) \longmapsto \mathcal{B}_\exists \subseteq \mathcal{P}(A^*)$$

e.g. $L_{\varphi(x)} \subseteq (A \times 2)^*$ changes to $L_{\exists x.\varphi(x)} \subseteq A^*$

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\updownarrow & & \updownarrow \\
X & \longmapsto & \mathcal{M}(X, S)
\end{array}$$

Finitely additive measures valued in semiring S:

1. $\mu(\emptyset) = 0_S, \mu(X) = 1_S$,
2. $A \cap B = \emptyset$ implies
   $\mu(A \cup B) = \mu(A) + \mu(B)$

Finite Model Theory

- Fails: compactness, Craig's interpolation property, etc.
- Survives: Ehrenfeucht–Fraïssé games, HPT
- New: 0–1 laws, structural limits, comonadic constructions, etc.

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Structural limits [Nešetřil, Ossona de Mendez]

For a formula $\varphi(x_1, \ldots, x_n)$ and a finite σ-structure A,

$$\langle \varphi, A \rangle \;=\; \frac{|\{\, \bar{a} \in A^n \mid A \models \varphi(\bar{a}) \,\}|}{|A|^n} \qquad \textit{(Stone pairing)}$$

Mapping $A \mapsto \langle -, A \rangle$ defines an embedding

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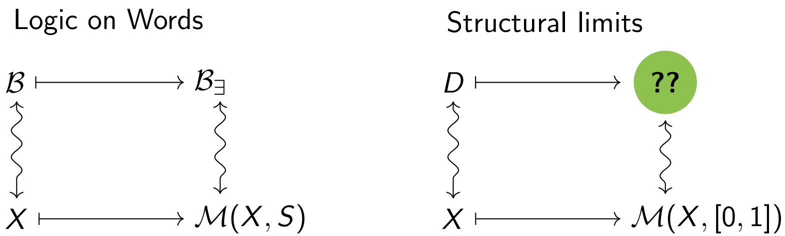
The limit of $(A_i)_i$ is computed as $\lim_{i \to \infty} \langle -, A \rangle$ in $\mathcal{M}(X_{\mathrm{FO}}, [0,1])$.

Are there any connections?



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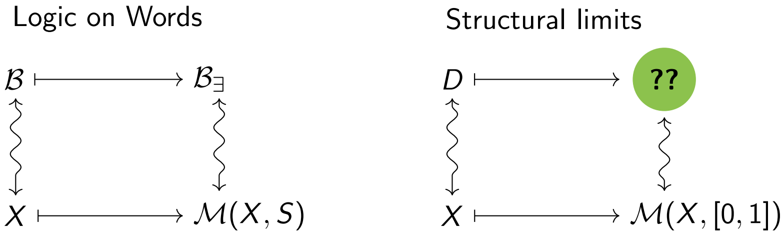

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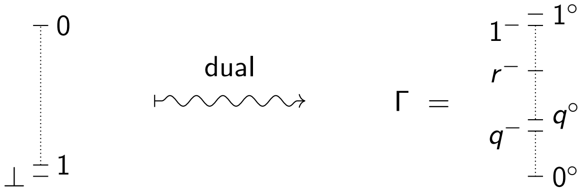
Our solution:

- Double the rationals in [0,1] to get a Priestley space Γ
- Then $\mathcal{M}(X, \Gamma)$ is also a Priestley space $\Longrightarrow$ has a dual

The space $(\Gamma, -, \sim)$

Define Γ as the dual of $([0,1] \cap \mathbb{Q}) < \{\top\}$ reversed:



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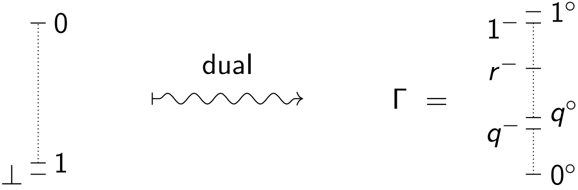

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- Semicontinuous partial operations $-$ and $\sim$ on Γ
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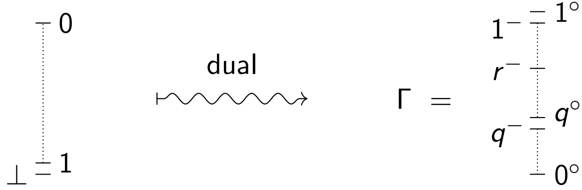

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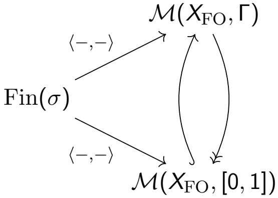

The dual of $X \mapsto \mathcal{M}(X, \Gamma)$

Given D, define $\mathbf{P}(D)$ as the *Lindenbaum-Tarski algebra* for the positive propositional logic on variables

$$\mathbb{P}_{\geq q}\,\varphi \qquad (\text{for } \varphi \in D, q \in [0,1] \cap \mathbb{Q})$$

and satisfying the rules

(L1) $p \leq q$ implies $\mathbb{P}_{\geq q}\,\varphi \models \mathbb{P}_{\geq p}\,\varphi$

(L2) $\varphi \leq \psi$ implies $\mathbb{P}_{\geq q}\,\varphi \models \mathbb{P}_{\geq q}\,\psi$

(L3) $\mathbb{P}_{\geq p}\,\mathbf{f} \models$ for $p > 0$, $\models \mathbb{P}_{\geq 0}\,\mathbf{f}$, and $\models \mathbb{P}_{\geq q}\,\mathbf{t}$

(L4) $\mathbb{P}_{\geq p}\,\varphi \wedge \mathbb{P}_{\geq q}\,\psi \models \mathbb{P}_{\geq p+q-r}\,(\varphi \vee \psi) \vee \mathbb{P}_{\geq r}\,(\varphi \wedge \psi)$ whenever $0 \leq p+q-r \leq 1$

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Theorem

If $D \leftrightsquigarrow X$ then $\mathbf{P}(D) \leftrightsquigarrow \mathcal{M}(X, \Gamma)$.

Logical reading of $\mathbf{P}(\mathcal{LT}_{\mathrm{FO}})$

Recall the embedding $\mathrm{Fin}(\sigma) \hookrightarrow \mathcal{M}(X_{\mathrm{FO}}, \Gamma)$, $A \mapsto \langle -, A \rangle$, where

$$\langle \varphi, A \rangle = \text{"\textit{the probability that a random assignment satisfies} } \varphi\text{"}$$

The duality $\mathbf{P}(\mathcal{LT}_{\mathrm{FO}}) \leftrightsquigarrow \mathcal{M}(X_{\mathrm{FO}}, \Gamma)$ provides the semantics:

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$\mathbb{P}_{\geq q}$ is a quantifier that binds all free variables.

Remark: We can also add negations, then $\mathbb{P}_{<q}$ is $\neg\mathbb{P}_{\geq q}$.

Comparison with the Logic on Words

The embedding

$$\mathrm{Fin}(\sigma) \to \mathcal{M}(X_{\mathrm{FO}}, \Gamma), \quad A \mapsto \langle -, A \rangle : X_{\mathrm{FO}} \to \Gamma$$

also used in the logic on words, for $\mathcal{B} \subseteq \mathcal{P}((A \times 2)^*)$,

$$A^* \to \mathcal{M}(X_{\mathcal{B}}, S), \quad w \mapsto \langle -, w \rangle : X_{\mathcal{B}} \to S$$

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- The same constructions!
- It's an embedding into the space of types of an extended logic

Future work

1. Model theory and proof theory for $\mathbf{P}(\mathcal{LT}_{\mathrm{FO}})$
2. Nesting of quantifiers
3. Relate to the new comonadic approach to the Finite Model Theory, studied by Abramsky et al.

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Thank you for your attention!

(check out arXiv:1907.04036)