Resolving Two Paradoxes About Knowledge States in the Foundations of Intuitionistic Analysis

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The Questions I am Sure Most of you Have

› What is a knowledge state?

› How do they come into the foundations of Intuitionistic Analysis?

› What are these two paradoxes?

WHAT ARE KNOWLEDGE STATES?

1. Choice Sequences
   a) What are they?
   b) Why are they important?
   c) Lawless sequences
2. Knowledge States
   a) Finite information
   b) Construction
   c) Collections of

Choice sequences : μ, ν

Choice Sequences – What are they?

"Choice sequences are functions of type $\mathbb{N} \to \mathbb{N}$ whose inner workings may not be entirely lawlike, i.e. not governed by an algebraically expressed function"

– Appleby 2017

Examples:

- Die rolls
- The function $\lambda x. 2x$

Can never be treated as "completed" objects. Only a finite amount of information is known (important!).

WHAT ARE KNOWLEDGE STATES?

Choice sequences : μ, ν

Choice Sequences – Why are they important?

Brouwer (1918a) introduced choice sequences to bridge the gap between $\mathbb{Q}$ and $\mathbb{R}$.

They are formally used in Kleene and Vesley (1965) to found intuitionistic analysis.

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Choice sequences : μ, ν

Choice Sequences – Lawless sequences

Lawless sequences are simply choice sequences where the generating process is entirely unknown.

First formalised in Kreisel (1968), and then refined in Troelstra (1977).

More on these later!

WHAT ARE KNOWLEDGE STATES?

Knowledge States: σ, σ',…

Knowledge States – Finite information

A knowledge state is a collection of information.

"σ is consistent with μ" – $\sigma(\mu)$ iff

1. σ is intensional information about μ
2. σ is extensional observations of elements of μ
3. σ is some combination of both

$SE(w,x,y)$ – "The x^{th} element of the w^{th} sequence in our list is y"

WHAT ARE KNOWLEDGE STATES?

Knowledge States: σ, σ',…

Tupality: $|\sigma|$

Knowledge States – Construction

Atomic knowledge states – Individual facts or elements

All knowledge states are formed by conjuncting ($\sqcap$), or disjuncting ($\sqcup$) existing knowledge states.

The number of sequences mentioned in a knowledge state is its “tupality”.

Quick Order – $\sigma \sqcup \sigma' \leq \sigma \leq \sigma \sqcap \sigma'$

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Knowledge States: σ, σ',…

Tupality: $|\sigma|$

Knowledge States – Collections of

Σ_{SE} – Knowledge states that are just lists of elements. (no intensional information)

Σ_T – Knowledge states that contain no contradictions (i.e. ones we have shown are consistent with some choice sequence)

Σ – The universe of all knowledge states.

THE FOUNDATIONS OF INTUITIONISTIC ANALYSIS

$\mathbb{R} \longrightarrow \mathbb{R}$ – Choice sequences as reals

Each choice sequence "represents" an element of $\mathbb{R}$.

Whenever we talk of a real, we are actually talking about a choice sequence.

Equality – $\mu = \nu \leftrightarrow \forall x[\mu(x) = \nu(x)]$

THE FOUNDATIONS OF INTUITIONISTIC ANALYSIS

$$\mu = \nu \leftrightarrow \forall x[\mu(x) = \nu(x)]$$
$$\mu = \nu \rightarrow \Psi(\mu) = \Psi(\nu)$$

$\mathbb{R} \rightarrow \mathbb{R}$ – Extensionality

(Total) Continuous operations (Ψ) of type $\mathbb{R} \rightarrow \mathbb{R}$ are **extensional**.

Extensionality – $\mu = \nu \rightarrow \Psi(\mu) = \Psi(\nu)$

THE FOUNDATIONS OF INTUITIONISTIC ANALYSIS

$$\mu = \nu \leftrightarrow \forall x[\mu(x) = \nu(x)]$$
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$\mathbb{R} \longrightarrow \mathbb{R}$ – Neighbourhood Functions

Can only work with finite information about choice sequences.

Neighbourhood functions (*e*) represent continuous operations, and are of type $\Sigma \rightarrow \mathbb{N}$ (sufficient for what we want)

Key facts:- **Always evaluated**, **stable**, **knowledge modulus of one**. (These are all axiomatically enforced)

THE FOUNDATIONS OF INTUITIONISTIC ANALYSIS

$\mu = \nu \leftrightarrow \forall x[\mu(x) = \nu(x)]$

Goal

Show that all continuous operations of type $\mathbb{R} \to \mathbb{R}$ can be represented by neighbourhood functions which only require finite lists of elements to be evaluated.

Essentially:

$$\forall \Psi \exists e \, \forall \mu \in \mathbb{R} \, \exists \sigma \in \Sigma_{SE} [\Psi(\mu) = e(\sigma)]$$

THE FOUNDATIONS OF INTUITIONISTIC ANALYSIS

Axioms and Definitions – Knowledge Axioms

Con – 1 $\forall\sigma\forall\mu[\sigma(\mu) \to \forall\sigma' \leq \sigma[\sigma'(\mu)]]$

Con – 2 $\forall\sigma\forall\mu[\sigma(\mu) \vee \neg\sigma(\mu)]$

AX-MOD – $\forall\sigma\forall\mu[\sigma(\mu) \to |\sigma| \leq 1]$

All these are specifically given in Appleby (2017), save Con – 2, which is something new that we would **REALLY** like to keep.

Axioms and Definitions – Lawless Sequences II

A choice sequence is (knowledge state) lawless (M_{KSLS}) **iff** the only knowledge that may be possessed about it of arity 1 is knowledge in Σ_{SE}.

$$\mu \in M_{KSLS} \leftrightarrow \forall^1 \sigma[\sigma(\mu) \to \sigma \in \Sigma_{SE}]$$

THE FOUNDATIONS OF INTUITIONISTIC ANALYSIS

1. $\mathbb{R} \to \mathbb{R}$
   a) Choice sequences as reals
   b) Extensionality
   c) Neighbourhood functions
2. Goal
3. Axioms and Definitions
   a) Knowledge axioms
   b) Lawless sequences II
4. Path to our Goal

$$\mu = \nu \leftrightarrow \forall x[\mu(x) = \nu(x)]$$

$$\forall^i \sigma \leftrightarrow \forall \sigma_{|\sigma|=i}$$

$$\mu \in M_{KSLS} \leftrightarrow \forall^1 \sigma[\sigma(\mu) \to \sigma \in \Sigma_{SE}]$$

NH1 $\forall \mu \exists \sigma[\sigma(\mu) \wedge \mathrm{e}(\sigma) \in \mathbb{N}]$

NH2 $\forall \sigma \forall \sigma' [\sigma \leq \sigma' \to e(\sigma) \leq \mathrm{e}(\sigma')]$

Path to our Goal

1. Given any Ψ, there exists an e representing $\Psi(\mu)$, for any given μ.
2. Take $\nu \in M_{KSLS}$ such that $\mu = \nu$
3. e only has $\sigma \in \Sigma_{SE}$ [definition of $\nu \in M_{KSLS}$] to work with when evaluating Ψ for ν, and it has to give an answer, so we know $\exists \sigma \in \Sigma_{SE}$[e($\sigma$) evaluates]. [NH1 and NH2]
4. Since $\mu = \nu$, $\sigma(\mu)$, since σ is just a list of elements. [definition of equality]
5. We know σ is enough to evaluate e. Hence

$$\forall \Psi \exists e\, \forall \mu \in \mathbb{R}\, \exists \sigma \in \Sigma_{SE}[\Psi(\mu) = e(\sigma)]$$

THE TWO PARADOXES

Con – 1

$\forall\sigma\forall\mu[\sigma(\mu) \to \forall\sigma' \leq \sigma[\sigma'(\mu)]]$

AX-MOD

$\forall\sigma\forall\mu[\sigma(\mu) \to |\sigma| \leq 1]$

Quick Order

$\sigma \sqcup \sigma' \leq \sigma \leq \sigma \sqcap \sigma'$

First Paradox – Bad axioms

Take any σ and any μ such that $\sigma(\mu)$

Take any σ' such that $|\sigma'|$>1

$\sigma \sqcup \sigma' \leq \sigma$ hence, by Con–1, $\sigma \sqcup \sigma'(\mu)$

But $|\sigma \sqcup \sigma'| > 1$

This clearly violates AX-MOD!

THE TWO PARADOXES

Con – 1

$$\forall\sigma\forall\mu[\sigma(\mu) \rightarrow \forall\sigma' \leq \sigma[\sigma'(\mu)]]$$

AX-MOD

$$\forall\sigma\forall\mu[\sigma(\mu) \rightarrow |\sigma| \leq 1]$$

Quick Order

$$\sigma\sqcup\sigma' \leq \sigma \leq \sigma\sqcap\sigma'$$

First Paradox – Two Options

1. Modify AX-MOD to give meaning to $\sigma\sqcup\sigma'(\mu)$.
2. Modify Con-1 to prevent it from being introduced.

(1) Either

a) Forces us to ignore information about sequences not present, which allows us to say nonsense about them.

b) Forces us to say such a sequence exists, which loses us Con-2, the property we REALLY wanted to keep.

THE TWO PARADOXES

1. First Paradox
   a) Bad axioms
   b) Two options
   c) The choice
2. Second Paradox
   a) No KS-lawless sequences!
   b) No path to analysis!
   c) New definition
   d) Restored path to analysis

$\mu \in M_{KSLS} \leftrightarrow \forall^1 \sigma[\sigma(\mu) \rightarrow \sigma \in \Sigma_{SE}]$

First Paradox – The choice

(2), on the other hand, has no such problems. Hence our solution to the first paradox is to modify Con-1 in the following way.

Con – 1*

$$\forall \sigma \forall \mu[\sigma(\mu) \rightarrow \forall^1 \sigma' \leq \sigma[\sigma'(\mu)]]$$

This change doesn't impact any of the existing results in the theory.

THE TWO PARADOXES

$$\mu \in M_{KSLS} \leftrightarrow \forall^1 \sigma[\sigma(\mu) \to \sigma \in \Sigma_{SE}]$$

Second Paradox – No KS-lawless sequences!

Take any $\mu \in M_{KSLS}$ and any σ such that we have $\sigma(\mu)$.

Take any $\sigma' \notin \Sigma_{SE}$ but still of modulo one.

Then, again $\sigma(\mu) \to \sigma \sqcup \sigma'(\mu)$ and $\sigma \sqcup \sigma' \notin \Sigma_{SE}$.

Hence, we have shown that M_{KSLS} is actually empty!

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~~$\mu \in M_{KSLS} \leftrightarrow \forall^1 \sigma[\sigma(\mu) \rightarrow \sigma \in \Sigma_{SE}]$~~

Second Paradox – No Path to Analysis!

(3) Take $\nu \in M_{KSLS}$ such that $\mu = \nu$

(4) e only has $\sigma \in \Sigma_{SE}$ [definition of $\nu \in M_{KSLS}$] to work with when evaluating evaluating Ψ for ν, and it has to give an answer, so we know $\exists\sigma \in \Sigma_{SE}$[e($\sigma$) evaluates]. [NH1 and NH2]

We need a new definition that gives

$$\forall\nu \in M_{KSLS} \exists\sigma \in \Sigma_{SE}[\sigma(\nu) \wedge e(\sigma) \in \mathbb{N}]$$

Second Paradox – New Definition

Any knowledge state consistent with a lawless sequence is weaker than a Σ_{SE} also consistent with said sequence.

$$\mu \in M_{KSLS} \leftrightarrow \forall^1 \sigma[\sigma(\mu) \rightarrow \exists^1 \sigma' \in \Sigma_{SE}[\sigma \leq \sigma' \wedge \sigma'(\mu)]]$$

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(3) Take $\nu \in M_{KSLS}$ such that $\mu = \nu$ [M_{KSLS} is no longer empty]

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Second Paradox – Restored path to analysis

(3) Take $\nu \in M_{KSLS}$ such that $\mu = \nu$ [M_{KSLS} is no longer empty]

(4a) We know there is a σ such that $\sigma(\nu)$ sufficient to evaluate e.

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(4a) We know there is a σ such that $\sigma(\nu)$ sufficient to evaluate e. [NH1]

(4b) We also know that there is a stronger, knowledge state in Σ_{SE}, consistent with ν [New definition]

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(4c) Is also sufficient to evaluate e. [NH2 and NH3]

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Our result restored!

The Answers to the questions

- What is a knowledge state?
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- What is a knowledge state? – A collection of finite information
- How do they come into the foundations of Intuitionistic Analysis? – They're part and parcel of choice sequences, a crucial tool for bridging the gap between $\mathbb{Q}$ and $\mathbb{R}$
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The Answers to the questions

- What is a knowledge state? – A collection of finite information
- How do they come into the foundations of Intuitionistic Analysis? – They're part and parcel of choice sequences, a crucial tool for bridging the gap between $\mathbb{Q}$ and $\mathbb{R}$
- What are these two paradoxes? – One was a badly formed axiom (Con-1), the other was a poor definition of M_{KSLS}. Both of them are now history!

Thanks For Listening

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- Special Thanks to Dr Peter Fletcher of Keele University

- **References**

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