Residuals and Conjugates in Positive Substructural Logics

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Some Stage Setting

As defined in Jónsson and Tarski's seminal *Boolean Algebras with Operators*, f_1, f_2 (unary operations on a Boolean algebra) are *conjugated* when:

$$a \wedge f_1 b = \bot \iff b \wedge f_2 a = \bot$$

This can be extended naturally to binary operations g_1, g_2, g_3:

$$a \wedge g_1(b, c) = \bot \iff b \wedge g_2(c, a) = \bot \iff c \wedge g_3(a, b) = \bot$$

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As has been studied, for instance by Jónsson and Tsinakis (1993), in the Boolean setting, relations of conjugation and *residuation* are, in some sense, equivalent.
Recall that unary h is residuated by h_1, and binary i by i_1, i_2 when:

$$ha \leq b \iff a \leq h_1 b$$
$$i(a, b) \leq c \iff a \leq i_1(b, c) \iff b \leq i_2(c, a)$$

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If a unary f is residuated, by f^r, then a conjugate is definable as $f^c a = \neg f^r \neg a$ – and similarly, $f^r a = \neg f^c \neg a$. This is true considering higher-arity operations

Some More Stage Setting

The logics discussed in this talk are presented as *binary assertional systems* or, if you prefer, *FMLA-FMLA* sequent systems – where the basic objects are pairs of formulae presented $A \vdash B$. All logics discussed here extend meet-semilattice logic (**MSL**), characterisable by the following axioms and rules:

- $A \vdash A$
- $A \wedge B \vdash A$, $A \wedge B \vdash B$

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C} \qquad\qquad \frac{A \vdash B \qquad A \vdash C}{A \vdash B \wedge C}$$

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Further, for the most part, we discuss logics extending distributive lattice logic (**DLL**), for which add the following axioms and rule:

- $A \vdash A \vee B$, $B \vdash A \vee B$
- $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

$$\frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C}$$

Even More Stage Setting

In either of these settings, extended by $\bot$ (with the additional axiom $\bot \vdash A$), conjugation between two unary operators $\Diamond_1, \Diamond_2$, or three binary operators $\circ_1, \circ_2, \circ_3$ amounts to the following rules:

$$\frac{A \wedge \Diamond_1 B \vdash \bot}{B \wedge \Diamond_2 A \vdash \bot} \qquad \begin{array}{c} A \wedge (B \circ_1 C) \vdash \bot \\ \hline\hline B \wedge (C \circ_2 A) \vdash \bot \\ \hline\hline C \wedge (A \circ_3 B) \vdash \bot \end{array}$$

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Whereas residuation, for which we'll reserve $\Box_i$ as the residual for $\Diamond_i$ $(1 \leq i \leq 2)$, and $\leftarrow_j, \rightarrow_j$ as residuals for $\circ_j$ $(1 \leq j \leq 3)$ is characterised by the following rules:

$$\frac{\Diamond_i A \vdash B}{A \vdash \Box_i B} \qquad \begin{array}{c} A \circ_j B \vdash C \\ \hline\hline A \vdash B \rightarrow_j C \\ \hline\hline B \vdash C \leftarrow_j A \end{array}$$

Aim of the Talk

The aim of the talk is to investigate the behaviour of conjugates (and residuals) in relational semantics for positive logics (by which we mean logics lacking Boolean negation and falsum).

The system with Lambek-style connectives (including conjugates) and Boolean negation has been investigated by Mikulás (1996), but the positive fragment appears not to have been studied, and presents some interesting difficulties.

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- Definability
- Canonicity
- Adequate Axiomatisations

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To begin, I'll briefly present the simple case of bimodal (temporal) positive logic – using tools developed by Dunn (1995).

Then I'll move to Routley-Meyer style semantics for logics with multiple families of Lambek-style connectives. First I'll discuss (briefly) why conjugates are interesting in that particular setting. Then we obtain some preliminary results leaving (frustratingly) open the question of finding an adequate axiomatisation for the full system.

$\mathcal{L}^{\mathcal{M}}$ is the language generated by a set of propositional variables $\mathbb{P}$ and connectives $\wedge, \vee, \Diamond_1, \Diamond_2, \Box_1, \Box_2$ of arities $2, 2, 1, 1, 1, 1$, respectively.

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$\mathcal{L}^{\mathcal{S}}$ generated by $\mathbb{P}$ and connectives $\wedge, \vee, \{\circ_i\}_{1 \leq i \leq 3}, \{\rightarrow_i\}_{1 \leq i \leq 3}, \{\leftarrow_i\}_{1 \leq i \leq 3}$ all of arity 2.

${\mathcal{L}^{\mathcal{S}}}_{[\wedge, \circ_i]}$ will be the subset of $\mathcal{L}^{\mathcal{S}}$ in the connectives $\wedge$ and $\{\circ_i\}$ (this is related to the Strictly Positive Modal logics studied by Kikot et. al. (2019)) – and ${\mathcal{L}^{\mathcal{S}}}_{[\wedge, \vee, \circ_i]}$ extends that by $\vee$ (and so on).

Definition

A frame F is a triple $\langle W, S_1, S_2 \rangle$ s.t. $W \neq \varnothing, S_1, S_2 \subseteq W^2$. Furthermore F is *2-cyclical* whenever $S_1 \alpha \beta \iff S_2 \beta \alpha$. A model M on F is a $V^M : \mathbb{P} \longrightarrow \wp(W)$, extended to $[\![\cdot]\!]^M : \mathcal{L}^{\mathcal{M}} \longrightarrow \wp(W)$ where:

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- $[\![p]\!]^M = V^M(p)$
- $[\![A \wedge B]\!]^M = [\![A]\!]^M \cap [\![B]\!]^M$
- $[\![A \vee B]\!]^M = [\![A]\!]^M \cup [\![B]\!]^M$
- $[\![\Diamond_i A]\!]^M = \{\alpha : \exists\beta(S_i\beta\alpha \ \& \ \beta \in [\![B]\!]^M)\}$
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Frame Semantics for a Basic Temporal Logic

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$A \vDash_M B \iff [\![A]\!]^M \subseteq [\![B]\!]^M$
$A \vDash_F B \iff A \vDash_M B$ for all M on F

While we (apparently) don't have the vocabulary to express conjugation between diamonds in the object language $\mathcal{L}^{\mathcal{M}}$, we can naturally express this at the level of models (or, if you prefer, at the level of the (full) complex algebra of the frame):

$$[\![A \wedge \Diamond_1 B]\!] = \varnothing \iff [\![\Diamond_2 A \wedge B]\!] = \varnothing$$

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$$[\![A \wedge \Diamond_1 B]\!] = \varnothing \iff [\![\Diamond_2 A \wedge B]\!] = \varnothing$$

Furthermore, if F has a 2-cyclical frame, then the above property is satisfied for any M on F.

Proposition

F is 2-cyclical iff $A \wedge \Diamond_i B \vDash_F \Diamond_i(\Diamond_j A \wedge B)$ $(i \neq j \in \{1,2\})$

An Adequate Axiomatisation

To the basic system of **DLL**, the following additions result in **TL** (as before $i \neq j \in \{1, 2\}$):

- $A \wedge \Diamond_i A \vdash \Diamond_i(\Diamond_j A \wedge B)$
- $\Box_i(A \vee B) \vdash \Box_i A \vee \Diamond_j B$

$$\frac{\Diamond_i A \vdash B}{\overline{A \vdash \Box_i B}}$$

$$\frac{A \vdash B}{\Box_i A \vdash \Box_i B} \qquad \frac{A \vdash B}{\Diamond_i A \vdash \Diamond_i B}$$

An Adequate Axiomatisation of **TL**

Theorem

$A \vdash B$ in **TL** $\iff$ for every 2-cyclical F, $A \vDash_F B$

The proof proceeds by the style of Dunn (1995). That is, a canonical model $\langle W^c, S_1^c, S_2^c, V^c \rangle$ is defined as usual, but that

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$$S_i^c \alpha\beta \iff \text{both (1) } A \in \beta \Rightarrow \Diamond_j A \in \alpha \text{ and (2) } \Box_i A \in \alpha \Rightarrow A \in \beta$$

For those interested, some consequences of the proof system sketched above which are used in the proof are as follows:

- $\Diamond_i \Box_i A \vdash A$, $A \vdash \Box_i \Diamond_i A$
- $\Box_i (A \wedge B) \dashv\vdash \Box_i A \wedge \Box_i B$
- $\Diamond_i (A \vee B) \dashv\vdash \Diamond_i A \vee \Diamond_i B$
- $\Diamond_i A \wedge \Box_j B \vdash \Diamond_i (A \wedge B)$

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- $\Diamond_i A \wedge \Box_j B \vdash \Diamond_i (A \wedge B)$

The argument is standard.

Now to the more interesting setting – $\mathcal{L}^S$.

Definition

$F = \langle W, \{R_i\}_{1 \leq i \leq 3} \rangle$ is a *frame* when $W \neq \varnothing, R_i \subseteq W^3$. Furthermore, F is *cyclical* when $R_1\beta\gamma\alpha \iff R_2\gamma\alpha\beta \iff R_3\alpha\beta\gamma$. A model M adds $V : \mathbb{P} \longrightarrow \wp(W)$, extended to $[\![\cdot]\!] : \mathcal{L} \longrightarrow \wp(W)$ as follows:

- Prop. variables, $\wedge, \vee$ as before
- $[\![B \circ_i C]\!]^M = \{\alpha : \exists\beta, \gamma(R_i\beta\gamma\alpha \;\&\; \beta \in [\![B]\!]^M \;\&\; \gamma \in [\![C]\!]^M)\}$
- $[\![B \rightarrow_i C]\!]^M = \{\alpha : \forall\beta, \gamma(R_i\alpha\beta\gamma \;\&\; \beta \in [\![B]\!]^M. \Rightarrow \gamma \in [\![C]\!]^M)\}$
- $[\![C \leftarrow_i B]\!]^M = \{\alpha : \forall\beta, \gamma(R_i\beta\alpha\gamma \;\&\; \beta \in [\![B]\!]^M. \Rightarrow \gamma \in [\![C]\!]^M)\}$

$A \vDash_M B$ iff $[\![A]\!]^M \subseteq [\![B]\!]^M$
$A \vDash_F B$ iff $\forall M$ on F, $A \vDash_M B$

Model Definability

Proposition

$A \wedge (B \circ_i C) \vDash_F (B \wedge (C \circ_{i+1} A)) \circ_i (C \wedge (A \circ_{i-1} B)) \iff F$ is cyclical.

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Also, when a frame is cyclical, we have:

$$[\![A \wedge (B \circ_i C)]\!] = \varnothing \iff [\![B \wedge (C \circ_{i+1} A)]\!] = \varnothing \iff [\![C \wedge (A \circ_{i-1} B)]\!] = \varnothing$$

Why is Conjugation Interesting Here?

According to one natural reading of the ternary relation of Routley-Meyer style semantics for relevant and substructural logics, it is a kind of dynamic update. On a dynamic reading of R, the usual truth conditions for $\rightarrow, \circ$ can be understood as saying that:

(1) α supports $A \rightarrow B$ whenever γ is a possible result of applying channel α to signal β, and β satisfies A only if γ satisfies B

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This reading naturally leaves room for other operators conjugating fusion. Let us set $\{\leftarrow, \circ, \rightarrow\}$ as $\{\leftarrow_1, \circ_1, \rightarrow_1\}$, and consider the truth conditions for their conjugates:

α supports $A \circ_2 B$ whenever it, as channel, can be applied to some A-supporting state, to obtain a B-supporting state.

α supports $A \circ_3 B$ whenever it is a signal for which some A-supporting state takes to a B-supporting state.

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α supports $A \circ_3 B$ whenever it is a signal for which some A-supporting state takes to a B-supporting state.

It seems a natural expressive extension of the language to provide this extra insight into the process being modeled by R.

A More Specific Motivation

Further on the dynamic reading, in a language including $\top$, there is a natural 'box' operator definable:

$$[\![\top \to_1 A]\!] = \{\alpha : \forall\gamma(\exists\beta(R_1\alpha\beta\gamma) \Rightarrow \gamma \in [\![A]\!])\}$$

Using techniques developed by Bimbó and Dunn (2008), an iterated version of this box is definable, so one can build a dynamic modal logic from the ternary relation semantics. However, the most natural 'diamond' for this operation, involves the conjugated $\circ_2$ – as $R_2\beta\gamma\alpha \iff R_1\alpha\beta\gamma$:

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It would be nice to develop a deeper understanding of the dynamic logic induced by the ternary relation semantics, and this motivates enriching the language with conjugation – one which doesn't require the (overly?) powerful expressive resources provided by Boolean negation.

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A more technical consideration: due to a result of Kurucz et al (1995), the full (associative) Lambek calculus with Boolean negation is undecidable, so one might hope that a more feasible system is available without negation.

An Attempt at Axiomatisation

Following the lead of the simple temporal case, it's reasonable to think that we can obtain the needed system by adding to **DLL** the following axioms and rules (where $1 \leq i \leq 3$ and $+, -$ are 'mod 2'):

- $A \wedge (B \circ_i C) \vdash (B \wedge (C \circ_{i+1} A)) \circ_i (C \wedge (A \circ_{i-1} B))$
- $A \rightarrow_i (B \vee C) \vdash (A \rightarrow_i B) \vee (A \circ_{i+1} C)$
- $(B \vee C) \leftarrow_i A \vdash (B \leftarrow_i A) \vee (C \circ_{i-1} A)$

$$\cfrac{\cfrac{A \circ_i B \vdash C}{A \vdash B \rightarrow_i C}}{B \vdash C \leftarrow_i A}$$

$$\frac{A \vdash B}{C \circ_i A \vdash C \circ_i B} \qquad \frac{A \vdash B}{A \circ_i C \vdash B \circ_i C}$$

Call the full system **LC** and let $\mathbf{LC}_{[\wedge,\circ_i]}$ be that including only rules and axioms mentioning formulae in $\mathcal{L}^{\mathcal{S}}{}_{[\wedge,\circ_i]}$.

It is easy enough to show soundness, as usual. But the usual completeness result is proving evasive. I'll sketch some stumbling blocks which seem to get in the way, but start with some preliminary results.

Completeness of $\mathbf{LC}_{[\wedge,\circ_i]}$

For $\mathbf{LC}_{[\wedge,\circ_i]}$, a completeness proof can be given in the style of Došen (1992) (as we may concern ourselves just with principal theories/filters), but just as in the modal case, we need to define the canonical relation with some care. Letting $W^C = \mathcal{L}_{[\wedge,\circ_i]}$, $V^C(p) = \{A : A \vdash p\}$, if we take the obvious route:

$$R_i^C BCA \iff A \vdash B \circ_i C$$

We cannot prove that $R_1^C BCA \iff R_2^C CAB \iff R_3^C ABC$ – if this were the case, the following rule would be valid:

$$\cfrac{\cfrac{A \vdash B \circ_1 C}{B \vdash C \circ_2 A}}{C \vdash A \circ_3 B}$$

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and this rule would permit the following disastrous derivation:

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(and, just to be clear, for $a, b, c \in \mathbb{P}$ there are models M on cyclical frames F where $a \nvDash_M b \circ_i c$ – constructing one such is left as an exercise for the interesting listener)

Proposition

$A \vdash_{\mathbf{LC}_{[\wedge,\circ_i]}} B \iff A \vDash_F B$ for all cyclical F

Instead, define the canonical accessibility relations, following Dunn (1995), as:

$$\begin{aligned} R_i^c BCA \text{ iff } & (1)\ A \vdash B \circ_i C \text{ and} \\ & (2)\ B \vdash C \circ_{i+1} A \text{ and} \\ & (3)\ C \vdash A \circ_{i-1} B \end{aligned}$$

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(2) $B \vdash C \circ_{i+1} A$ and
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...with this it is immediate that $\langle W^c, \{R_i^c\}\rangle$ is cyclical, and the truth lemma for $\circ_i$ primarily relies on the axiom defining cyclicality, for note that if $B \circ_1 C \in A$ then:

$$A \vdash A \wedge (B \circ_1 C) \vdash (B \wedge (C \circ_2 A)) \circ_1 (C \wedge (A \circ_3 B)) \ (1)$$

Proposition

$A \vdash_{\mathbf{LC}_{[\wedge,\circ_i]}} B \iff A \vDash_F B$ for all cyclical F

Instead, define the canonical accessibility relations, following Dunn (1995), as:

$$
\begin{aligned}
R_i^c BCA \text{ iff } & (1)\ A \vdash B \circ_i C \text{ and} \\
& (2)\ B \vdash C \circ_{i+1} A \text{ and} \\
& (3)\ C \vdash A \circ_{i-1} B
\end{aligned}
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$$A \vdash A \wedge (B \circ_1 C) \vdash (B \wedge (C \circ_2 A)) \circ_1 (C \wedge (A \circ_3 B)) \ (1)$$

From this, between monotonicity and the cyclicality axiom, the following result:

$$B \wedge (C \circ_2 A) \vdash (C \wedge (A \circ_3 B)) \circ_2 A \ (2)$$

$$C \wedge (A \circ_3 B) \vdash A \circ_3 (B \wedge (C \circ_2 A)) \ (3)$$

Thus $R_1^c(B \wedge (C \circ_2 A))(C \wedge (A \circ_3 B))A$, $B \wedge (C \circ_2 A) \vdash B$, and $C \wedge (A \circ_3 B) \vdash C$.

But it is not generally the case that:

$$R_i^c AB(A \circ_i B)$$

As neither of the following hold.

$$A \vdash B \circ_{i+1} (A \circ_i B)$$
$$B \vdash (A \circ_i B) \circ_{i-1} A$$

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So, in a similar construction for the full $\mathcal{L}^{\mathcal{S}}$, we don't generally have:

$$R_i^c (A \rightarrow_i B)AB$$
$$R_i^c A(B \leftarrow_i A)B$$

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which throws a wrench in extending this argument to $\mathbf{LC}_{[\wedge, \circ_i, \rightarrow_i, \leftarrow_i]}$. (Though one can extend the desired result to $\mathbf{LC}_{[\wedge, \vee, \circ_i]}$ by a Pair Extension argument)

Moving to Full **LC**

The most natural way to proceed is by setting W^c to be the set of all prime **LC**-theories, $V^c(p) = \{\alpha \in W^c : p \in \alpha\}$, and:

$$
\begin{aligned}
R_i^c\beta\gamma\alpha \text{ iff } & (1)\ D \in \beta \;\&\; E \in \gamma. \Rightarrow D \circ_i E \in \alpha \text{ and}\\
& (2)\ D \in \gamma \;\&\; E \in \alpha. \Rightarrow D \circ_{i+1} E \in \beta \text{ and}\\
& (3)\ D \in \alpha \;\&\; E \in \beta. \Rightarrow D \circ_{i-1} E \in \gamma
\end{aligned}
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Defining R^c_1, R^c_2, and R^c_3 separately (in the usual fashion), as in the situation with $\mathbf{LC}_{[\wedge,\circ_i]}$, does not allow the proof of circularity (so, cyclicality is not *canonical* w.r.t. **LC** – just as it is not w.r.t. $\mathbf{LC}_{[\wedge,\circ_i]}$). But using the above definition seems to put one in problems with the Truth Lemma.

It can be shown (in a more or less standard way) that:

$$B \circ_i C \in \alpha \iff \exists \beta, \gamma \in W^c (R^c_i \beta\gamma\alpha \ \&\ \beta \in B \ \&\ \gamma \in C)$$

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Defining R_1^c, R_2^c, and R_3^c separately (in the usual fashion), as in the situation with $\mathbf{LC}_{[\wedge,\circ_i]}$, does not allow the proof of circularity (so, cyclicality is not *canonical* w.r.t. **LC** – just as it is not w.r.t. $\mathbf{LC}_{[\wedge,\circ_i]}$). But using the above definition seems to put one in problems with the Truth Lemma.

It can be shown (in a more or less standard way) that:

$$B \circ_i C \in \alpha \iff \exists\beta,\gamma \in W^c(R_i^c\beta\gamma\alpha \;\&\; \beta \in B \;\&\; \gamma \in C)$$

But there seem to be difficulties with applying the Pair Extension lemma to keep out the bad guys – as is needed to employ the usual arguments for:

$$\forall\beta,\gamma \in W^c((R_i^c\alpha\beta\gamma \;\&\; B \in \beta) \Rightarrow C \in \gamma) \Rightarrow B \to_i C \in \alpha$$

It seems that the proposed additional axioms may not guarantee that when β', γ' (theories) are defined so that $R_i^c\alpha\beta'\gamma'$ (for a version of this relation the relata of which may be theories), $B \in \beta'$, $C \notin \gamma'$, these can be extended to prime β, γ satisfying the same properties.

The usual technique involves fixing γ first, then constructing β – but we seem to need to construct them in lock-step in order to ensure that all three conditions are satisfied, and we haven't found a construction that does everything required.

The obvious candidate sequents (which we thought of) for allowing the proof of the key lemmas are unsound. Nor have we found a counterexample to completeness. So:

Open Problem

Axiomatise the logic of cyclical frames in the full language $\mathcal{L}^{\mathcal{S}}$.

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Axiomatise the logic of cyclical frames in the full language $\mathcal{L}^{\mathcal{S}}$.

With that posed, I'll end by turning to some lingering considerations concerning extending the well known relevant substructural logics with conjugates.

There are a number of interesting correspondences in the extended language. For instance, the following is noteworthy:

$A \circ_2 B \vdash B \circ_2 A$ corresponds to $R_1\alpha\beta\gamma \Rightarrow R_1\alpha\gamma\beta$

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$A \circ_2 B \vdash B \circ_2 A$ corresponds to $R_1 \alpha\beta\gamma \Rightarrow R_1 \alpha\gamma\beta$

This is a postulate characteristic, in the usual setting, of adding Boolean negation to some substructural logic (for instance in **KR** (*corrupted* **R**) studied by Urquhart (1984), where the ternary relation fully permutes).

In the usual setting, this results from the interpretation of negation in terms of the star $* : W \longrightarrow W$:

$$[\![\neg A]\!] = \{\alpha : \alpha^* \in [\![A]\!]\}$$

along with the star condition corresponding to contraposition: $R\alpha\beta\gamma \Rightarrow R\alpha\gamma^*\beta^*$ and the Boolean condition $\alpha^* = \alpha$.

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along with the star condition corresponding to contraposition: $R\alpha\beta\gamma \Rightarrow R\alpha\gamma^*\beta^*$ and the Boolean condition $\alpha^* = \alpha$. I don't know of any natural positive formulae which correspond to these conditions without invoking conjugates.

The elements which provide for logics with *theorems* of the form of individual formulae, rather than sequents, on this semantics are a partial order $\sqsubseteq \subseteq W^2$ and a set of normal worlds $N \subseteq W$ – these are related such that $\alpha \sqsubseteq \beta \iff \exists \gamma \in N(R_1 \gamma \alpha \beta)$.

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To account for the order, we now have to define models a bit differently – in particular,
$\wp(W)^{\uparrow} = \{X \subseteq W : \alpha \sqsubseteq \beta \,\&\, \alpha \in X. \Rightarrow \beta \in X\}$

and extend the valuation clauses for complex formulae accordingly.

Simplified Semantics

To show that a model can be so extended to cover the full language we need the following tonicity conditions on R_i:

$$\text{If } \alpha' \sqsubseteq \alpha, \beta' \sqsubseteq \beta, \text{ and } \gamma \sqsubseteq \gamma', \text{ then } R_i\alpha\beta\gamma \text{ implies } R_i\alpha'\beta'\gamma'.$$

These conditions ensure that $\rightarrow_i$, $\leftarrow_i$, and $\circ_i$ are indeed operations on $\wp(W)^{\uparrow}$.

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These conditions ensure that $\rightarrow_i$, $\leftarrow_i$, and $\circ_i$ are indeed operations on $\wp(W)^{\uparrow}$.

Cyclicality is what tells us that R_1, R_2, R_3 are offering three different 'perspectives' on one and the same R – if cyclicality holds, then this R will need to have all of these tonicity properties at every point in the relation. That is, it will have to be monotone and antitone in each position – R will be *tonally overloaded*.

Simplified Semantics

If we define $\alpha \sqsubseteq \beta$ iff $\exists\gamma \in N(R_1\gamma\alpha\beta)$ and we assume these tonicity conditions then we obtain the result that $\alpha \sqsubseteq \beta \Rightarrow \alpha = \beta$.

So all models with conjugates and residuals in which the usual relationship between the ternary relation(s) and the set of normal worlds will satisfy the condition:

$$\text{If } \alpha \in N,\ R_1\alpha\beta\gamma \text{ iff } \beta = \gamma$$

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So all models with conjugates and residuals in which the usual relationship between the ternary relation(s) and the set of normal worlds will satisfy the condition:

$$\text{If } \alpha \in N,\ R_1\alpha\beta\gamma \text{ iff } \beta = \gamma$$

This is the characteristic condition of Priest and Sylvan's (1992) *Simplified Semantics* for Relevant logics – there, the above is the *only* frame condition put on R, and otherwise the frame is flat (ordered only by $=$). This is a complete semantics for **B**, and can be extended, as has been shown by Restall (1993) and Roy (2009) to cover other logics in the area.

So the extension of the logical vocabulary by conjugates and residuals provides (potentially) another way to motivate the simplified semantic framework.

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