On Scott's semantics for many-valued logic

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13/Aug/2019

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- It is used in [5] as a proof theory for many-valued logic.
- We shall point out a flaw in Scott's soundness proof for this.
- We shall offer a remedy for this, but observe this in turn affects Scott's completeness proof.

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- Let $\mathscr{A}, \mathscr{B} := \mathscr{A} \cup \mathscr{B}$; $\mathscr{A}, B := \mathscr{A} \cup \{B\}$ and $\mathscr{A} \vdash \mathscr{B} := (\mathscr{A}, \mathscr{B}) \in \vdash$.

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- One may understand commas on the left (right) hand side of $\vdash$ as conjunctions (disjunctions): e.g. $\mathscr{A}, A, B \vdash C, D, \mathscr{B}$

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Entailment relation satisfies the next three rules.

- (R) $\mathscr{A} \vdash \mathscr{B}$ if $\mathscr{A} \cap \mathscr{B} \neq \emptyset$ *reflexitivity*
- (M) $\dfrac{\mathscr{A} \vdash \mathscr{B}}{\mathscr{A}, \mathscr{A}' \vdash \mathscr{B}, \mathscr{B}'}$ *monotonicity*
- (T) $\dfrac{\mathscr{A} \vdash B, \mathscr{C} \qquad \mathscr{A}, B \vdash \mathscr{C}}{\mathscr{A} \vdash \mathscr{C}}$ *transitivity*

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Example (Negri et al. 2004 [2])

Let $S = X \times X$. Then the entailment relation of total quasi order is generated by axioms $\vdash (a, a)$; $(a, b), (b, c) \vdash (a, c)$ and $\vdash (a, b), (b, a)$.

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Example (Rinaldi et al. 2018 [4])

Let S be a commutative ring. Then the entailment relation of prime filter on S is generated by axioms $\vdash 1$; $a, b \vdash ab$; $ab \vdash a$; $a + b \vdash a, b$ and $0 \vdash$.

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- Let $\mathscr{S}$ be a set closed under connectives $\wedge, \vee, \rightarrow$.
- Let $I = [0, e)$ be a half-open interval of an ordered abelian group.
- We shall consider a family of valuations $\mathscr{V} = \{\mathfrak{v}_i : i \in I\}$ where $\mathfrak{v}_i : \mathscr{S} \rightarrow \{\mathbf{t}, \mathbf{f}\}$.

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(or) $\mathfrak{v}_i(A \vee B) = \mathbf{t} \Leftrightarrow \mathfrak{v}_i(A) = \mathbf{t}$ or $\mathfrak{v}_i(B) = \mathbf{t}$.

(if) $\mathfrak{v}_i(A \rightarrow B) = \mathbf{t} \Leftrightarrow$ whenever $i + j \leq k$ and $\mathfrak{v}_j(A) = \mathbf{t}$, then $\mathfrak{v}_k(B) = \mathbf{t}$.

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(if) $\mathfrak{v}_i(A \to B) = \mathbf{t} \Leftrightarrow$ whenever $i + j \leq k$ and $\mathfrak{v}_j(A) = \mathbf{t}$, then $\mathfrak{v}_k(B) = \mathbf{t}$.

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- Furthermore, $\mathfrak{v}_i$ satisfies the conditions:

(con) if $i \leq j$ and $\mathfrak{v}_i(A) = \mathbf{t}$, then $\mathfrak{v}_j(A) = \mathbf{t}$.

(min) if $\mathfrak{v}_i(A) = \mathbf{t}$ holds for some i, then there is a minimal $m \in I$ such that $\mathfrak{v}_m(A) = \mathbf{t}$.

Many Valuations

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- Write $\mathscr{A} \Vdash_{\mathfrak{v}} \mathscr{B}$ when $\mathfrak{v}(A) = \mathbf{t}$ for all $A \in \mathscr{A}$ implies $\mathfrak{v}(B) = \mathbf{t}$ for some $B \in \mathscr{B}$.
- Write $\mathscr{A} \vdash_{\mathscr{V}} \mathscr{B}$ if $\mathscr{A} \Vdash_{\mathfrak{v}_i} \mathscr{B}$ for all $i \in I$.

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$$(\wedge)\quad \frac{\mathscr{A}, A, B \vdash \mathscr{B}}{\mathscr{A}, A \wedge B \vdash \mathscr{B}}$$

$$(\vee)\quad \frac{\mathscr{A} \vdash A, B, \mathscr{B}}{\mathscr{A} \vdash A \vee B, \mathscr{B}}$$

$$(\to_1)\quad \frac{A, \mathscr{B} \vdash \mathscr{C}}{A \to \mathscr{B} \vdash A \to \mathscr{C}}\ [\mathscr{C} \neq \emptyset]$$

$$(\to_2)\quad \frac{\mathscr{C} \vdash \mathscr{B}, A}{\mathscr{B} \to A \vdash \mathscr{C} \to A}\ [\mathscr{C} \neq \emptyset]$$

$$(\to_3)\quad A \to (B \to C) \vdash B \to (A \to C)$$

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$\vdash$ is claimed to be sound and complete with $\vdash_{\mathscr{V}}$, i.e. $\mathscr{A} \vdash \mathscr{B} \Leftrightarrow \mathscr{A} \vdash_{\mathscr{V}} \mathscr{B}$.

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- We show a derivable rule, (Ex) $\dfrac{A \vdash B \to C}{B \vdash A \to C}$

$$
\dfrac{\dfrac{\dfrac{A \vdash B \to C}{\vdash A \to (B \to C)}\,(\to_1)}{\vdash A \to (B \to C), B \to (A \to C)}\,(\text{M}) \qquad \dfrac{}{A \to (B \to C) \vdash B \to (A \to C)}\,(\to_3)}{\dfrac{\vdash B \to (A \to C)}{B \vdash A \to C}\,(\to_1)}\,(\text{T})
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- (Scott 1974 [5]) claims the rules are sound, but does not show the case:
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- Using $(\to_2)$, we can prove $(A \to B) \to B \vdash A, B$.

$$\cfrac{\cfrac{\cfrac{}{(A \to B) \to B \vdash (A \to B) \to B}\ (\mathrm{R})}{A \to B \vdash ((A \to B) \to B) \to B}\ (\mathrm{Ex})}{(A \to B) \to B \vdash A, B}\ (\to_2)$$

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- However $\mathfrak{v}_0(A) = \mathfrak{v}_0(B) = \mathbf{f}$, so $(A \to B) \to B \nVdash_{\mathfrak{v}_0} A, B$.
- Therefore the entailment relation is not sound with $\vdash_{\mathscr{V}}$.

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- For soundness we need to add the condition:

  (min') For any A there is a minimal i s.t. $\mathfrak{v}_i(A) = \mathbf{t}$.
- In fact, (Urquhart 2001 [8]) already considers a similar semantics for $\mathbf{L}_\omega$, but with (min') assumed.

  ($\mathbf{L}_\omega$ corresponds to an instance of Scott's system where $\mathscr{S}$ is a set of formulas.)

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- But this $\varepsilon(A)$ need not be finite, in which case (min') is violated.

Hence, Scott's system <u>still requires a completeness proof.</u>

Acknowledgements

Acknowledgements

This research was carried out within a project of the "Core-to-Core Program (A. Advanced Research Networks)" funded by **the Japan Society for the Promotion of Science (JSPS)**, as well as the project "A New Dawn of Intuitionism: Mathematical and Philosophical Advances" (ID 60842) funded by **the John Templeton Foundation.** (*The opinions expressed in this talk are those of the authors and do not necessarily reflect the views of any of the recognized foundations.*) A part of this research took place at **the Hausdorff Research Institute for Mathematics**, Bonn, during its Trimester Program "Types, Sets and Constructions". The related financial support is gratefully acknowledged.

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