Non-monotonic abstract multiset consequence relations

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Substructural logics

= LK/LJ minus some structural rules (optionally: plus some axioms)

- Weakening: $\Gamma_1, \Gamma_2 \Rightarrow \Delta \;/\; \Gamma, A, \Gamma_2 \Rightarrow \Delta$
- Contraction: $\Gamma_1, A, A, \Gamma_2 \Rightarrow \Delta \;/\; \Gamma_1, A, \Gamma_2 \Rightarrow \Delta$
- Exchange: $\Gamma_1, A, \Gamma_2, B, \Gamma_3 \Rightarrow \Delta \;/\; \Gamma_1, B, \Gamma_2, A, \Gamma_3 \Rightarrow \Delta$

(and similarly on the right-hand side)

= Logics of (various classes of) residuated lattices

- Include: Lambek calculus, relevant, linear, and fuzzy logics, ...
- Interpretation: categorial grammar, possible-world semantics, degrees of truth, formulae-as-resources, ...

Tarski consequence relation $\vdash \subseteq \mathcal{P}(\mathsf{Fm}_\mathcal{L}) \times \mathsf{Fm}_\mathcal{L}$:

1. Reflexivity: If $\varphi \in X$ then $X \vdash \varphi$
2. Monotonicity: If $X \vdash \varphi$ and $X \subseteq Y$, then $Y \vdash \varphi$
3. Cut: If $Y \vdash \varphi$ and $X \vdash \psi$ for all $\psi \in Y$, then $X \vdash \varphi$
4. (Finitarity, substitution invariance)

Operates on sets of premises

- $\Rightarrow$ Presupposes the structural rules
- $\Rightarrow$ Can only represent the external consequence relation of substructural logics = preservation of designated values

The internal consequence in substructural logics (representing the validity of substructural implication) requires a non-Tarskian relation, with sequences or (assuming exchange) multisets of premises

Multiset consequence relations:

- Avron (1992)
  single-conclusion, without weakening
- Cintula–Paoli, Cintula–Gil-Férez–Moraschini–Paoli (2019)
  multiple-conclusion, with weakening
- Běhounek–Cintula–Lávička, (this talk, in progress)
  multiple-conclusion, without weakening

Why:

- Some logics have no single-conclusion presentation
  (eg, Łukasiewicz: $[p \otimes q] \vdash [p, q]$ non-representable)
- To include weakening-free logics (relevant, uninorm fuzzy, $\mathrm{FL_e}$, ...)
  (relevance, degrees of full truth, negative resources, ...)
- Assuming exchange (simpler, still reasonably broad)

Non-monotonic multiset deductive relation:

(finite multiset $\vdash$ finite multiset, $\otimes$-conjunctive reading on both sides)

1. Reflexivity: $\Gamma \vdash \Gamma$
2. Transitivity: If $\Gamma \vdash \Delta$ and $\Delta \vdash \Pi$, then $\Gamma \vdash \Pi$
3. Compatibility: If $\Gamma \vdash \Delta$, then $\Gamma, \Pi \vdash \Delta, \Pi$ (resource separability)

Cf multi-conclusion adaptation of Avron's simple consequence relation:

1. Simple reflexivity: $\varphi \vdash \varphi$
2. Finitary cut: If $\Gamma \vdash \Delta, \varphi$ and $\Gamma', \varphi \vdash \Delta'$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$
3. Combining: If $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$ (optional)

Observation: refl + comp $\Longleftrightarrow$ refl + comb

Variants of cut:

1. If $\Gamma \vdash \Pi$ and $\Pi, \Gamma' \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta$
2. If $\Gamma \vdash \Delta, \Pi$ and $\Gamma', \Pi \vdash \Delta'$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$

Observation:

$$\mathsf{refl} + \mathsf{cut}_1 \Longrightarrow \mathsf{trans} + \mathsf{comp} \Longrightarrow \mathsf{cut}_2 \Longrightarrow \mathsf{cut}_1 \Longrightarrow \mathsf{cut}_{\mathsf{finitary}}$$

Corollary:

- Every non-monotonic multiset consequence relation is Avron's (multi-conclusion) simple consequence relation
- Non-monotonic multiset consequence relations can equivalently be defined by Reflexivity and $\text{Cut}_{(1 \text{ or } 2)}$

Abstract non-monotonic consequence relations (Blok–Jónsson–style):

- Abstract objects instead of multisets of formulas
- Finite multisets show the structure a of dually integral Abelian pomonoid $\mathbf{M} = (M, \leq, +, 0)$
  (for $\leq$ multiset inclusion, $+$ multiset union, 0 the empty multiset)

Definition

An abstract non-monotonic consequence relation on a dually integral Abelian pomonoid $\mathbf{M} = (M, \leq, +, 0)$ is a relation $\vdash$ on M such that:

1. $a \vdash a$ (Reflexivity)
2. If $a \vdash b$ and $b \vdash c$, then $a \vdash c$ (Transitivity)
3. If $a \vdash b$, then $a + c \vdash b + c$ (Compatibility)

- Finitarity expressible by means of compact elements of $\mathbf{M}$
- Substitution-invariance expressible as invariance wrt monoidal actions
- $\vdash$ is a compatible preorder on $\mathbf{M}$

Deductively closed theories

In Tarski consequence relations, a deductive closure of $X \subseteq \mathsf{Fm}_{\mathcal{L}}$ is the largest set Y st $X \vdash Y$, so an *element* of $\mathcal{P}(\mathsf{Fm}_{\mathcal{L}})$

In multiset consequence relations, the largest multiset need not exist
Eg, often $\Gamma \vdash \Delta$ and $\Gamma \vdash \Pi$, but $\Gamma \nvdash \Delta \vee \Pi$ in Łukasiewicz logic:

Let $\Gamma = [p, q, p \leftrightarrow q]$, then $\Gamma \vdash [p, p]$ and $\Gamma \vdash [q, q]$, but $\Gamma \nvdash [p, p, q, q]$

$\Rightarrow$ As the deductive closure of a multiset Γ we take the set of all consequences of Γ, so a subset of M

Definition: A deductively closed theory in $\vdash$ on $\mathbf{M}$ is any $\vdash$-upset of M

Observation: The family $\mathsf{Th}(\vdash)$ of all $\vdash$-theories is a closure system on M

Denote:

- $\mathsf{Th}_{\vdash}(X)$ the smallest $\vdash$-theory containing $X \subseteq M$
- $\mathsf{Th}^{\mathsf{p}}(\vdash)$ the set of *principal* $\vdash$-theories of the form $\mathsf{Th}_{\vdash}(a)$

  = the set of all principal $\vdash$-upsets

Proposition:

1. Each theory is a union of principal theories
2. $\mathsf{Th}_{\vdash}(X) = \bigcup_{x \in X} \mathsf{Th}_{\vdash}(x)$

Theorem:

For $\vdash$ on $\mathbf{M}$ define $+_{\vdash}$ on $\mathsf{Th}^{p}(\vdash)$: $\quad \mathsf{Th}_{\vdash}(x) +_{\vdash} \mathsf{Th}_{\vdash}(y) = \mathsf{Th}_{\vdash}(x + y)$

Then: $\quad \mathbf{Th}^{\mathsf{p}}_{\vdash} = \big(\mathsf{Th}^{\mathsf{p}}(\vdash), \subseteq, +_{\vdash}, \mathsf{Th}(0)\big)$ is a dually integral Abelian pomonoid and the mapping $\mathsf{Th}_{\vdash} \colon \mathsf{M} \to \mathsf{Th}^{p}(\vdash)$ is a surjective morphism