Bernoulli disjointness

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A *subflow* of X is any non-empty, closed, G-invariant $Y \subseteq X$. We say X is *minimal* if the only subflow of X is X itself. Equivalently, X is minimal if every orbit is dense. By Zorn's lemma, every flow contains a minimal subflow.

Definition

Let X and Y be G-flows. We say that X and Y are *disjoint* if whenever $Z \subseteq X \times Y$ is a subflow such that $\pi_X[Z] = X$ and $\pi_Y[Z] = Y$, we have $Z = X \times Y$. This is written $X \perp Y$.

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Example: $G = \mathbb{Z}$, X is irrational rotation by α, Y is irrational rotation by β with α/β irrational.

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Furstenberg (1969): for $G = \mathbb{Z}$, we have $X \perp A^{\mathbb{Z}}$ for every minimal flow X.

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2. *For any finite symmetric $D \subseteq G$, there is a D-spaced set $S \subseteq G$ so that for any $x \in X$, $Sx \subseteq X$ is dense (SDOP).*
3. *For any finite symmetric $D \subseteq G$ and any $U \subseteq X$, there is a D-spaced set $S \subseteq G$ so that $S^{-1}U = X$ (SCP).*

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Theorem (Glasner-Tsankov-Weiss-Z.)

Let G be an infinite discrete group. Then G has BDJ.

Step 1 - A G-flow X is called *essentially free* if for any $g \in G \setminus \{1_G\}$, the set $\{x \in X : gx = x\}$ is nowhere dense. To show that G has BDJ, it suffices to show that $X \perp 2^G$ for some essentially free minimal flow X.

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Step 2 - For a fixed essentially free flow X, saying that X has the SCP is suitably first order. So it suffices to prove that every countable group has BDJ.

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We show that minimal "suitably equicontinuous" flows have BDJ. So G has BDJ whenever G admits an infinite, normal, maxap subgroup H.

Step 4 - A G-flow X is called *proximal* if for every $x, y \in X$, there is a net $g_i \in G$ and some $z \in X$ with $(g_i x, g_i y) \to (z, z)$. We show that every minimal proximal flow has the BDJ.

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Recently, Frisch, Tamuz, and Vahidi-Ferdowsi have shown that every countably ICC group acts *faithfully* on some minimal proximal flow. They ask about getting *free* minimal proximal flows.

Theorem (Glasner-Tsankov-Weiss-Z.)

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Idea: getting essentially free is actually quite easy given what Frisch, Tamuz, and Vahidi-Ferdowsi do. Then one can use a *highly proximal extension* to turn essentially free into free while preserving both minimality and proximality.

Step 5 - If G is an infinite group, $H = G/F$ for $F \trianglelefteq G$ some finite normal subgroup, and H has the BDJ, then so does G.

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Remark - Strangely enough, the only way we know how to prove this step is to work directly with $M(G)$ and $M(H)$.

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If Z is finite, consider $F/Z \trianglelefteq G/Z$. The group F/Z is an infinite, residually finite, normal subgroup, so we are done by steps 3 and 5.

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A G-flow X has the *trapping property* if for every non-empty open U, there is $n < \omega$ so that for any distinct $g_0, ..., g_{n-1} \in G$, the set $g_0 U \cup \cdots \cup g_{n-1} U$ contains an orbit.

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Bernshteyn shows that every flow with the trapping property is disjoint from every minimal flow. Then he uses the Lovász Local Lemma (LLL) to show that 2^G has the trapping property.

The work of Frisch, Tamuz, and Vahidi-Ferdowsi makes use of the space of *strongly irreducible* subshifts of A^G. A subshift $X \subseteq A^G$ is strongly irreducible if there is a finite symmetric $D \subseteq G$ so that for any $x_0, x_1 \in X$ and any sets $S_0, S_1 \subseteq G$ which are D-apart, there is $y \in X$ with $y|_{S_i} = x_i|_{S_i}$.

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Let $\mathcal{S} \subseteq K(A^G)$ be the closure of the non-trivial strongly irreducible shifts. This space is particularly well suited to Baire category arguments. FTVF show that the minimal flows are dense G_δ in $\mathcal{S}$.

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One can show that for any BDJ group, every minimal flow is disjoint from every strongly irreducible subshift. Using the fact that disjointness from a given metrizable flow is a G_δ condition, we obtain:

Theorem

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Theorem (Glasner-Tsankov-Weiss-Z.)

Let G be a countably infinite group. Then there is a family $\{X_i : i < \mathfrak{c}\}$ of free minimal flows such that $\prod_{i<\mathfrak{c}} X_i$ is minimal.

As a corollary, we can compute the underlying space of $M(G)$ for any countably infinite group G.

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As G is countable $\pi w(M(G)) \leq \mathfrak{c}$. To show $\pi w(M(G)) \geq \mathfrak{c}$, it suffices to find some minimal flow attaining this value. Balcar and Błaszczyk do this for $\mathbb{Z}$, and Turek does this for abelian groups.

Theorem (Glasner-Tsankov-Weiss-Z.)

Let G be any countably infinite group. Then as a topological space, we have $M(G) \cong \mathrm{Gl}(2^{\mathfrak{c}})$.

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This is simply because for the minimal flow $X = \prod_{i<\mathfrak{c}} X_i$, we have $\pi w(X) = \mathfrak{c}$.

For each G-flow X, the *enveloping semigroup* of X is the set $E(X) := \mathrm{cl}_{pw}\{x \to gx : g \in G\} \subseteq X^X$. Under composition, $E(X)$ becomes a compact right topological semigroup.

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Let βG denote the space of ultrafilters on G. This is also a compact right-topological semigroup. Furthermore, we have a continuous map $\phi_X \colon \beta G \to E(X)$, where for $p \in \beta G$, we set $\phi_X(p)(x) = \lim_{g \to p} gx$.

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Glasner and Weiss (1983): no for $G = \mathbb{Z}$, drawing heavily from earlier work of Furstenberg (1969).
Conjecture of Pestov (1998): no for any non-precompact topological group G.

Theorem (Glasner-Tsankov-Weiss-Z.)

Let G be any infinite discrete group. Then $\phi_{M(G)}\colon \beta G \to E(M(G))$ is not an isomorphism.

Thanks!