Squares and Uncountably Singularized Cardinals

Maxwell Levine

Kurt Gödel Research Center
Universität Wien

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Joint work with Dima Sinapova!
Why singular cardinals?
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Theorem (Easton)

The continuum function $\kappa \mapsto 2^\kappa$ on regular cardinals is constrained only by:

- $\lambda \leq \kappa = \Rightarrow 2^\lambda \leq 2^\kappa$ (monotonicity)
- $\text{cf}(2^\kappa) > \kappa$ (König's Theorem).

– The question was: can we extend this to singular cardinals?
– No!
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GCH cannot fail for the first time at a singular of uncountable cofinality.

– The crux is that we can use Fodor’s Lemma.

Theorem (Magidor)
Relative to a supercompact cardinal, it is consistent that GCH holds below $\aleph_\omega$ but $2^{\aleph_\omega} > \aleph_{\omega+1}$.

Theorem (Shelah)
If $\aleph_\omega$ is a strong limit (in particular if GCH holds below $\aleph_\omega$) then $2^{\aleph_\omega} < \aleph_{\omega^4}$.
Singulars of countable versus uncountable cofinality

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Singularizing cardinals

Theorem (Prikry) If $\kappa$ is measurable in some ground model $V$, then there is a $\kappa^+$-c.c. forcing $P$ such that in $V\langle P\rangle$, $\kappa$ is a singular strong limit cardinal of countable cofinality.

Theorem (Silver) If $\kappa$ is supercompact, then it is consistent that there is a model where $\kappa$ is measurable and $2^{\kappa} > \kappa^+$.

Silver + Prikry $\Rightarrow$ \text{Con}(\kappa \text{ is a singular strong limit } \land 2^{\kappa} > \kappa^+)$

Theorem (Gitik) If there is a model of set theory in which a cardinal $\kappa$ is a singular strong limit and $2^{\kappa} > \kappa$, then this implies the consistency of large cardinals. This can be done so that the large cardinal assumption is optimal.
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But what actually *happens* when we singularize cardinals?
Defining squares

Definition

We say that $\langle C_\alpha | \alpha \in \lim(\kappa + \lambda) \rangle$ is a $\square_{\kappa, \lambda}$-sequence if for all limit $\alpha < \kappa + \lambda$:

1. each $C \in C_\alpha$ is a club subset of $\alpha$ with $\text{ot}(C) \leq \kappa$;
2. for every $C \in C_\alpha$, if $\beta \in \lim(C)$, then $C \cap \beta \in C_\beta$;
3. $1 \leq |C_\alpha| \leq \lambda$.

The $\square_{\kappa, \lambda}$-sequence cannot have a thread, i.e. there is no club $D \subset \kappa + \lambda$ such that $\forall \alpha \in \lim(D), D \cap \alpha \in C_\alpha$.

Note that $\square_{\kappa, 1}$ is just the original Jensen's $\square_\kappa$, and $\square_{\kappa, \kappa}$ is the weak square $\square_\kappa^\ast$. Also, ZFC proves $\square_{\kappa, 2\kappa}$.

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Uses of squares

Squares help us think about different models of set theory.

\[ \square \kappa \text{ for all } \kappa. \]

\[ \square \kappa,\lambda \text{ serves as a "yardstick" comparing a given model to } L, \text{ where a smaller } \lambda \text{ means a stronger resemblance to } L. \]

Square sequences can be used to show that large cardinals are necessary for certain results.

Squares also have specific combinatorial entailments.

\[ \text{GCH } + \square \kappa \text{ implies that there is a } \kappa^+\text{-Suslin tree.} \]

\[ \square^* \kappa \text{ is equivalent to a special } \kappa^+\text{-Aronszajn tree.} \]

If \( \mu < \kappa \) are infinite and \( \square \kappa \) holds, then \((\kappa^+, \kappa) \rightarrow (\mu^+, \mu)\) fails.
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- If $\mu < \kappa$ are infinite and $\Box_\kappa$ holds, then $\left( \kappa^+, \kappa \right) \not\rightarrow \left( \mu^+, \mu \right)$ fails.
Consequences of singularizing cardinals

Theorem (Džamonja-Shelah and Gitik)
Suppose there are models $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{W}$ and a cardinal $\kappa$ such that:

1. $\mathcal{M} \models \kappa$ is inaccessible",
2. $(\kappa +)_{\mathcal{M}} = (\kappa +)_{\mathcal{W}}$,
3. and $(\text{cf } \kappa)_{\mathcal{W}} = \omega$.

Then $\Box \kappa, \omega$ holds in $\mathcal{W}$.

Fact (Gitik-Sharon)
Assuming large cardinals, there are models $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{W}$ and a cardinal $\kappa$ such that:

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The main theorem

What happens if $\kappa$ is singularized to have uncountable cofinality?

Theorem (L.-Sinapova)
Assuming large cardinals, there are models $\mathcal{M} \subset V \subset W$ such that:

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- $(\kappa^+,\mathcal{M}) \models (\kappa^+,W),
- \omega < (\text{cf } \kappa)^W < \kappa,
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Outline of the proof: a description of the model
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It is known that if $\mu$ is a Mahlo cardinal and $\text{Col}(\kappa, < \mu)$ is the Lévy Collapse for making $\mu$ into $\kappa^+$, then $\square_{\kappa, \tau}$ fails in $V[\text{Col}(\kappa, < \mu)]$ for all $\tau < \kappa$. 
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- Begin with a model $\bar{\mathcal{V}}$ in which $\kappa$ is supercompact, $\mu$ is a Mahlo cardinal, and $\kappa < \mu$. 
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- In $\bar{V}[\text{Col}(\kappa, < \mu)]$, let $\mathbb{M}$ be Magidor’s variation of Prikry forcing for singularizing $\kappa$ to have an uncountable cofinality $\lambda$. 
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- Let $\text{Col}(\kappa, < \mu)$ be the Lévy Collapse for making $\mu$ into $\kappa^+$.
- In $\bar{V}[\text{Col}(\kappa, < \mu)]$, let $\mathbb{M}$ be Magidor’s variation of Prikry forcing for singularizing $\kappa$ to have an uncountable cofinality $\lambda$.
- Then the statement of the theorem holds if $\mathcal{V} = \bar{V}[\text{Col}(\kappa, < \mu)]$ and $\mathcal{W} = \bar{V}[\text{Col}(\kappa, < \mu) \ast \mathbb{M}]$. 
Outline of the proof: pointing to the technical crux

Fix $\tau < \kappa$. We want to show that $\square^{\kappa, \tau}$ fails in $W$. The steps are the following:

▶ Suppose for contradiction that $W$ has a $\square^{\kappa, \tau}$-sequence $C$.

▶ Argue that there is a model $V'$ such that $V \subset V' \subset W$ such that $V'$ has a $\square^{\kappa, \tau}$-sequence $C'$.

▶ Moreover, argue that $C'$ is not a $\square^{\kappa, \tau}$-sequence in $W$ because it has a thread $T$.

(This thread needs limit points, which is why this only works if $\kappa$ has uncountable cofinality.)

▶ The crux is to argue that $T$ could not have been added in the quotient $W/V'$.

The most important technical ingredient is the Prikry Density Lemma for Magidor Forcing.
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▶ Argue that there is a model \( V' \) such that \( V \subset V' \subset W \) such that \( V' \) has a \( \Box_{\kappa, \tau} \)-sequence \( C' \).

▶ Moreover, argue that \( C' \) is not a \( \Box_{\kappa, \tau} \)-sequence in \( W \) because it has a thread \( T \).

(This thread needs limit points, which is why this only works if \( \kappa \) has uncountable cofinality.)

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The most important technical ingredient is the Prikry Density Lemma for Magidor Forcing.
Further directions

Question: Does Magidor's forcing add a $\Box_{\kappa, <\kappa}$-sequence?

Suppose $\text{ON} \subset V \subset W$ are class models and $\kappa$ is a cardinal in $V$ such that:

1. $V \models \kappa$ is inaccessible
2. $W \models \text{cf} \kappa < \kappa$
3. $(\kappa^+)_{V} = (\kappa^+)_{W}$

Is there necessarily a $\Box_{\kappa, <\kappa}$-sequence?
Further directions

Question

Does Magidor’s forcing add a \( \square_{\kappa,<\kappa} \)-sequence?
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1. $V \models " \kappa \text{ is inaccessible}"$ ;
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Děkuji!