Stationary Reflection and Prikry type forcings

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Forcing and Independence

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More specifically, let $\mathbb{P} \in \mathrm{V}$ be a partial order and let $G \subseteq \mathbb{P}$ to be an filter. Moreover, we require that for every dense subset $D \subseteq \mathbb{P}$ from V, $G \cap D$ is non-empty. In this case, we say that G is a V-*generic filter*.

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It is easy to define from a generic filter G a sequence of κ many different reals (i.e. elements of 2^ω), so in $\mathrm{V}[G]$, $2^{\aleph_0} \geq \kappa$.

It is far less trivial to show that V and $\mathrm{V}[G]$ have the same cardinals, and thus by taking $\kappa = \aleph_2^{\mathrm{V}}$, one obtains a model of the failure of the continuum hypothesis.

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Moreover the following is open:

Question

Is it consistent for some regular cardinal $\lambda > \omega_1$, that there are no λ-Suslin trees in V but there is a λ-Suslin tree in $\mathrm{V}[G]$, where G is a Cohen real?

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A cardinal κ is regular if $\operatorname{cf} \kappa = \kappa$. Otherwise, we say that κ is singular.

Singular Cardinal Hypothesis

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In fact, the first method which was used in order to obtain the consistency of $2^\lambda > \lambda^+$ for a strong limit singular cardinal λ was to start with a regular cardinal λ, force with a variant of Easton's forcing in order to get $2^\lambda > \lambda^+$ and then force with *Prikry Forcing* in order to change the cofinality of λ to ω.

Measurable cardinals and Prikry Forcing

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Definition

A cardinal κ is called *measurable* if there is a κ-complete non-principle ultrafilter U on κ.

This is a large cardinal axiom.

Iterated Ultrapowers and Prikry Generic

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Using Łoš theorem, the map $j_1 \colon \mathrm{V} \to \mathrm{M}_1$ sending x to $[c_x]_U$ is an elementary embedding. One can verify that the critical point of j_1 (the least ordinal that moves) is κ.

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We obtain a sequence of models M_n and commutative system of elementary embeddings $j_{n,m}\colon \mathrm{M}_n \to \mathrm{M}_m$ for every $n < m$.

Let M_ω be the direct limit of this system and let κ_n be the critical point of the n-th elementary embedding. Let $\kappa_\omega = \sup \kappa_n$.

Basic Properties of $\mathbb{M}_\omega$

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Let $P = \langle \kappa_n \mid n < \omega \rangle$.

Theorem (Bukovský-Dehornoy)

$\mathrm{M}_\omega[P] = \bigcap_{n<\omega} \mathrm{M}_n$. *Moreover,* $\mathrm{M}_\omega[P]$ *is the generic extension of* M_ω *by a generic filter for the Prikry forcing.*

Prikry Forcing without forcing

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Lemma

Every bounded subset of κ_ω from $\mathrm{M}_\omega[P]$ belongs to M_ω.
κ_ω is a regular cardinal in M_ω and singular in $\mathrm{M}_\omega[P]$.

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For a regular and uncountable cardinal κ, the intersection of $< \kappa$ many clubs at κ is a club at κ.

Definition

A set $S \subseteq \alpha$ is *stationary* if it intersects every club at α.

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Does the same hold for stationary sets?

Definition

A stationary set S at κ *reflects* if there is an ordinal $\alpha < \kappa$ such that $S \cap \alpha$ is stationary at α.

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- More generally, given a successor cardinal κ^+, where κ is regular,
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is a stationary set that does not reflect.

On the other hand, it is consistent (relative to large cardinals), that those are the only limitations:

Stationary Reflection at the successor of a regular cardinal

Theorem (Harrington-Shelah)

There is a model of ZFC *in which every stationary subset of* $S^{\omega_2}_{\omega}$ *reflects if and only if there is a model of* ZFC *with a Mahlo cardinal.*

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This is an equiconsistency result.

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Theorem (Magidor)

Let μ be a singular limit of supercompact cardinals. Then every stationary subset of μ^+ reflects. Moreover, there is a forcing extension in which $\mu = \aleph_\omega$, and every stationary subset of $\aleph_{\omega+1}$ reflects.

Stationary reflection at the successor of a measurable

Fact

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Fact

Let κ be a measurable cardinal in V and let P be a generic filter for the Prikry forcing. Then $\left(S^{\kappa^+}_{\kappa}\right)^{\mathrm{V}}$ is stationary and non-reflecting in $\mathrm{V}[P]$.

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We will give a different proof for this theorem, which will use much of the ideals from the proof of the main theorem.

Proof

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Let us show that this holds for the extension of M_ω. Let $S \in \mathrm{M}_\omega[P]$ be a stationary subset of $\left(S^{\kappa_\omega^+}_{<\kappa_\omega}\right)^{\mathrm{M}_\omega}$.

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This set is also a member of M_n. If S_n is stationary in M_n then it reflects (by elementarity), and one can push this reflection to M_ω. Otherwise, for each n we have a club $C_n \in \mathrm{M}_n$ disjoint from S_n.

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Killing the bad stationary set

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As before, we would like to analyze this over $\mathrm{M}_\omega[P]$.

Fact

The forcing for shooting a club through $\left(S^{\kappa_\omega^+}_{<\kappa_\omega}\right)^{\mathrm{M}_\omega}$ in $\mathrm{M}_\omega[P]$ is equivalent in V to the Cohen forcing $\mathrm{Add}(\kappa^+, 1)$.

Let H be a V-generic filter for this forcing and work in $\mathrm{V}[H]$.

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Stationary reflection in $\mathrm{M}_\omega[\mathcal{H}]$

Theorem

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- Using a more complicated construction, we can collapse cardinals and get stationary reflection at $\aleph_{\omega+1}$.

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- Using a more complicated construction, we can collapse cardinals and get stationary reflection at $\aleph_{\omega+1}$.
- The precise large cardinal axiom which we use is slightly below κ being κ^+-supercompact.

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Otherwise, we have a club $C_n \in \mathrm{M}_n[\mathcal{H}_n]$ disjoint from S_n. The elementary embedding $j_{n,\omega}$ extends to an embedding:

$$\hat{j}_{n,\omega} \colon \mathrm{M}_n[\mathcal{H}_n] \to \mathrm{M}_\omega[\mathcal{H} \restriction n+1] \subseteq \mathrm{M}_\omega[\mathcal{H}].$$

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So $\hat{C}_n = \hat{j}_{n,\omega}(C_n) \in \mathrm{M}_\omega[\mathcal{H}]$ and $\bigcap \hat{C}_n$ is a club disjoint from S. □

Remarks

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- The model $\mathrm{M}_\omega[\mathcal{H}]$ is a forcing extension of M_ω using a Prikry type forcing. Yet, it is easier to analyze its properties without referring to the forcing notion.
- (joint with Ben-Neria and Unger) A similar, but more complicated application of the same ideas allow one to get also failure of SCH together with stationary reflection.
- While similar characterizations exist for Magidor forcing, we don't know how to get stationary reflection at the successor of a singular cardinal of uncountable cofinality with less than a limit of supercompact cardinals.

Questions

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- Can we use this approach in order to lower the consistency strength of stationary reflection at $\aleph_{\omega_1+1}$?
- It is possible to reduce the consistency strength of simultaneous reflection of infinite collection of stationary subsets of $\aleph_{\omega+1}$?

Thank You!

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