On Euclid's Notion of Equal Figures

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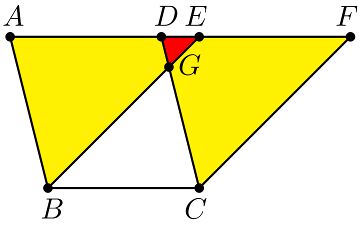
No area in Euclid. Instead, "equal figures"

- The word *area* never occurs in Euclid's *Elements*, despite the fact that area is clearly a fundamental notion in geometry.
- Instead, Euclid speaks of "equal figures."
- Apparently a "figure" is a simply connected polygon, or perhaps its interior.

What is a figure?

- Does it have an interior? or is it just made of lines?
- Does it have to be a polygon, or can it have curved boundaries?
- Does a circle count as a figure? An ellipse?
- Does it have to be convex?

Euclid's proof of Prop. I.35, parallelograms in the same parallels with the same base are equal.



Euclid wants to prove the parallelograms $ABCD$ and $BCFE$ are equal. He proves the triangles ABE and DCF are congruent. Implicitly, he assumes DEG and DGE are equal figures. Then "subtracting equals from equals", the yellow quadrilaterals are equal. Then, "adding equals to equals", he adds triangle BCG (implicitly assuming BCG is equal to BGC) to arrive at the desired conclusion.

Euclid's common notions

- Things which are equal to the same thing are also equal to one another.
- If equals be added to equals, the wholes are equal.
- If equals be subtracted from equals, the remainders are equal.
- Things which coincide with one another are equal to one another.
- The whole is greater than the part.

Euclid uses only the first three with figures.

He also uses "halves of equals are equals" with no justification whatever.

Equal figures are not $=$

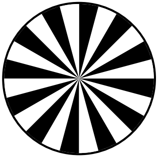
- $a = b$ means a can be substituted *anywhere* for b.
- That is not true of “equal figures” or “equal lines” for that matter.
- We use “congruence” for lines instead of $=$.
- What shall we use for equal figures?

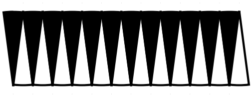
Can we use area to define "equal figures"?

- How about, two figures are "equal" if they have the same area?
- But then we would need to define area.
- Euclid probably knew that he did not know how to define area.
- So he appealed to the common notions about "equal figures".
- But that hasn't passed muster since at least 1882, now that we understand "equality" better, and have some idea of the difficulties in adding and subtracting and even defining regions.

Area in Greek mathematics

- The Greeks never multiplied lengths by lengths to get lengths.
- A length times a length is a rectangle.
- Multiply three lengths to get a solid.
- Of course engineers and architects calculated with areas. But areas were numbers.
- Heron of Alexandria had no problem multiplying four numbers to get an area in his famous triangle formula. But not a length.
- When Archimedes calculated the area of a circle, he didn't mention the word area. Instead he showed a circle is equal to a certain rectangle





Possible ways to define area or "equal area"

- Equidecomposition (cutting into triangles).
- Integral calculus
- Axiomatize the properties of area instead of defining it.

These options all involve non-geometric objects, such as natural numbers or real numbers. We also consider:

- Descartes's geometric arithmetic. Define multiplication by proportions and prove its properties by Desargues's and Pascal's theorems.

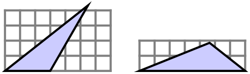
Segment arithmetic was emphasized by Hilbert and Tarski in their well-known formalizations of geometry. But it *does not* lead to a definition of "equal area".

Area of a triangle in elementary school

Teacher: The area of a triangle is half the base times the height.

Student: OK. But then why do I get the same answer after turning the triangle so a different side is the base? The height and width are different. For example if the sides are $(3, 5, 7)$:

$3 \cdot 4.33/2 = 6.49$



$7 \cdot 1.86/2 = 6.49$

Teacher: It's because the area is a property of the triangle itself, independent of the orientation.

Student: How do you know that?

Teacher : By the theorem on the change of variable in a double integral under a transformation of the domain of integration. See Apostol volume 2, pp. 88-89. The Jacobian of a rotation is 1.

Student: Oh, yes. Of course. Now I understand.

Half the base times the height won't work as a definition.

Multiplication, area, and equal figures

- Being able to multiply doesn't enable you to define area.
- Area wasn't defined until double integrals were understood.
- Therefore segment arithmetic doesn't enable you to define area.
- Neither Hilbert nor Tarski ever wrote about Euclid's notion of equal figures, as far as I know.
- Thus neither Hilbert nor Tarski proved I.35, or Euclid's version of the Pythagorean theorem, in their axiom systems.
- Their systems proved $c^2 = a^2 + b^2$, which is different. Euclid showed how to cut up the three squares into equal pieces.

Proof-checking Euclid

In 2017-2018, Narboux, Wiedijk, and I computer-checked proofs of about 235 propositions, including the 48 propositions of Euclid Book I. Our aims were

- To use axioms as close as possible to Euclid's.
- To ensure that all proofs were absolutely correct, by computer proof-checking.
- To use proofs as close to Euclid's as possible, while correcting gaps and errors.

To do this, we introduced 16 axioms about equal figures. We were not the first to do that: Hartshorne's textbook does something similar, and deZolt before him.

Actually, we introduced axioms about `ET` (equal triangles) and `EF` (equal quadrilaterals), which is enough for Book I.

Sixteen Axioms for Equal Figures

- Four cut axioms. If we cut equal triangles off of equal triangles [or quadrilaterals] getting triangles the results are equal. Also if we get quadrilaterals, the results are equal.
- Two paste axioms. If we paste equal triangles onto equal triangles [or quadrilaterals] getting triangles the results are equal. Also if we get quadrilaterals, the results are equal.
- Congruent triangles are equal.
- `halvesofequals`. If equal quadrilaterals are each divided into equal triangles by their diagonals, then all four triangles are equal.
- `ET` and `EF` are symmetric and transitive.
- `ETpermutation`. ABC is equal to BCA.
- `EFpermutation`. $ABCD$ is equal to $BCDA$, $DCBA$, etc.
- Two de Zolt axioms. If we cut a triangle off a triangle getting a triangle, then the original and new triangles are not equal.

Axioms and Intuition

I was not satisfied with axiomatizing “equal figures”.

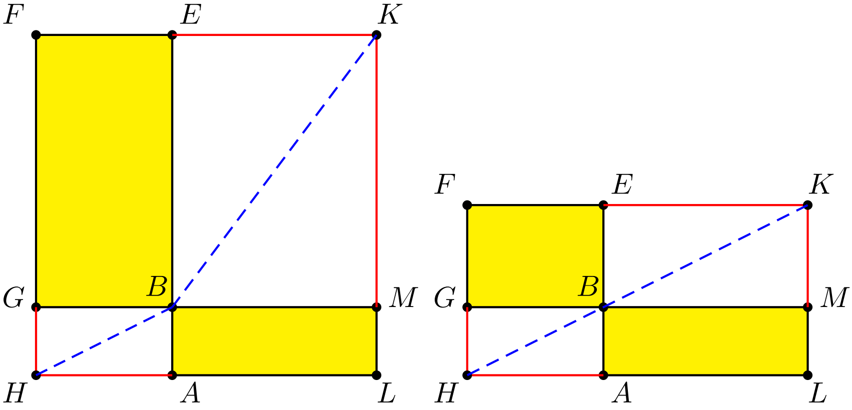
- Axioms are supposed to correspond to intuition and express evident truths.
- The notion of “equal figures” only makes intuitive sense as “equal areas”, but we can’t define or axiomatize area in the style of Euclid.
- The axioms for “equal figures” are *ad hoc*.
- Our first version was even inconsistent, due to ambiguities about whether figures have to be convex or not. Connecting two vertices of a non-convex polygon might not divide it into two parts.
- Besides, there’s nothing like equal figures in the famous axiomatizations of Hilbert and Tarski, which are widely supposed to be the modern replacement for Euclid.

So how can we avoid axioms for equal figures?

- We can't define area without integral calculus.
- But maybe we *can* define "equal figures".
- Perhaps even in way the Euclid might have done.

Yes! That can be done. That is the work I will now describe.

Definition of Equal Rectangles

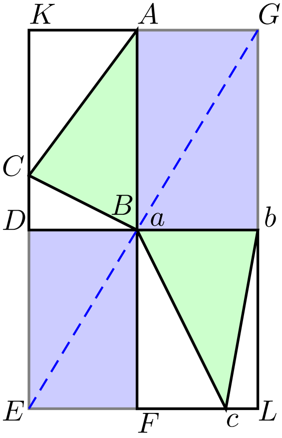


Place copies of two given rectangles as $BEFG$ and *BMLA* with two sides collinear. The other sides (extended) meet by Euclid 5 forming a large rectangle. Then $BEFG$ and *BMLA* are defined to be equal if $\mathbf{B}(H, B, K)$. (Left, unequal. Right, equal.)

Euclid I.44

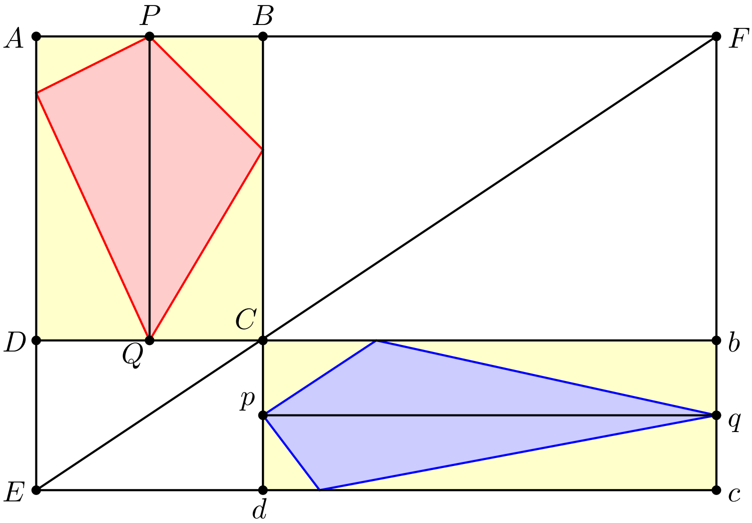
- The definition of equal rectangles has the same picture as Euclid I.44, except I.44 has a parallelogram, not just a rectangle.
- Hence, the definition is definitely "in the spirit of Euclid."
- Euclid could have given this definition.

Definition of Equal Triangles



ABC is equal to abc if EBG is a straight line.
C lies on line KD, not necessarily between K and D.

Definition of Equal Quadrilaterals



Use rectangles parallel to (either) diagonal.

What counts as a quadrilateral

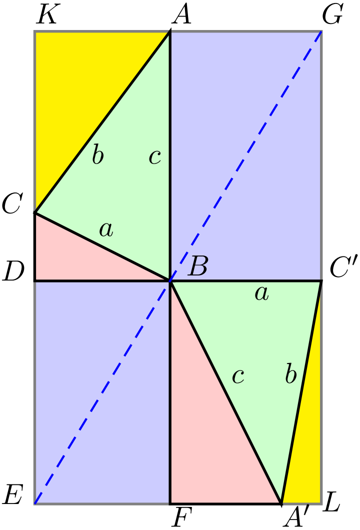
- Either convex (diagonals meet at a point between the vertices), or
- "really a triangle" (one vertex between the two adjacent).
- Non-convex quadrilaterals do not occur in Euclid Books I-IV.
- Convex quadrilaterals definitely lie in a plane because of the meeting diagonals.

The task at hand

- Then we “just” have to verify that all the 16 equal-figures axioms hold using these definitions.
- That turned out to be non-trivial.
- But in the end, we succeeded.

Verification that ABC and BCA are equal

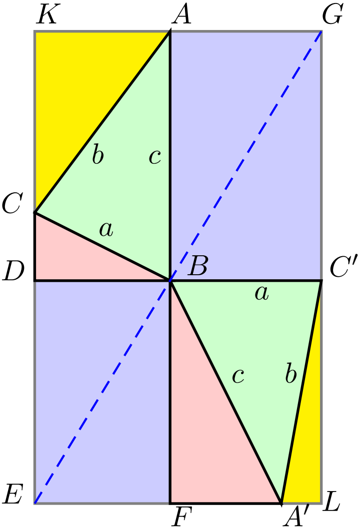
- The green triangles have equal angles at B.
- Therefore the pink triangles have equal angles at B.
- The pink triangles are right triangles, so their sides are proportional.
- Therefore $BF : EF = c : a$.
- Therefore triangles BEF and GBC' are similar.
- Therefore EBG is a straight line.



Area in school, take two

Student: Why do I get the same answer when I take a different side as the base?

Teacher: Let's see how that works out. Look at this picture:



Base time height is the same, whichever side is the base

- We have just given a proof, using only elementary geometry, of that fact.
- The only proof I knew before used double integrals.
- Note that the statement does not mention the word "area."
- To state it in geometry, we have to first define segment arithmetic.
- Since Hilbert and Tarski did that, the proof can be given in their systems.
- There had to be *some* proof, by known meta-theorems, but that proof might require the completeness axioms (about filling Dedekind cuts).
- Whether the definition of area as a double integral can be given in geometry we do not know. Not directly, since integrals involve natural numbers and limits.

Proportionality

- We need proportionality to verify $ABC = BCA$.
- Also to verify that "equal rectangles" is transitive.
- Rectangle with (width, height) (a, b) is equal to (c, d) iff

$$b : c = d : a$$

- That focuses attention on a Euclidean theory of proportions.
- All the other things we need to prove about equal triangles, equal rectangles, and equal quadrilaterals require only propositions from Book I before equal figures.

Proportionality in Euclid

- Proportion is treated in Book VI.
- It depends on Eudoxes's theory of "magnitudes" in Book V.
- That in turn depends on Archimedes's axiom, which we don't want to use.
- Besides, Prop. VI.2 uses the principle $ABC = BCA$, again (as in I.35) without explicit mention. Hence, trying to use Euclid's Book VI to verify $ABC = BCA$ would be circular.
- Therefore Book VI is irrelevant for our purposes.

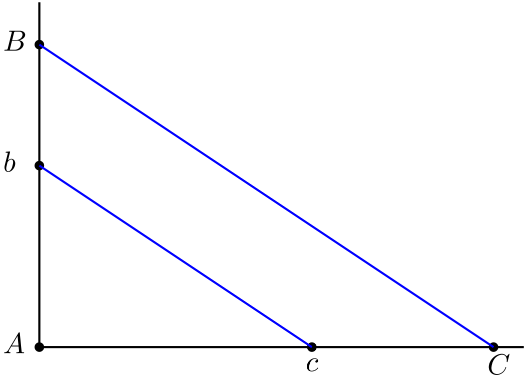
Early proportion theory

- It turns out we are far from the first to be interested in developing the theory of proportions by simpler means.
- This has been worked on by Hessenburg, Kupffer, Schur, Dehn, Hilbert, and finally Bernays, over the time period from 1893 to 1956.
- Bernays and Kupffer proved everything we need, by means similar to those used in Euclid Book I before equal figures are introduced.
- Bernays wrote Supplement II for the 8th edition of Hilbert's *Foundations of Geometry* (1956). He called it *A simplified development of the theory of proportion.*
- In §24 of Hilbert's book, Hilbert treated proportion using segment multiplication, which we don't want to use. But Bernays and Kupffer used only early Euclidean methods.

Kupffer

- Karl Kupffer gave two proofs of the "interchange theorem" that $a : b = c : d$ if and only if $a : c = b : d$ in 1893.
- In 1902, Schur wrote a letter to the *Mathematische Annalen* giving Kupffer's proofs (with credit).
- Kupffer's first proof is in Bernays, credited not to Kupffer but to F. Enrique. That turns out to be incorrect as in Enrique's book, Kupffer is cited.
- That chapter of Enrique's book is written by another author, not Enrique, and that author says Kupffer's first proof is actually due to Weierstrass, but gives no citation, so this has "fake news" status, it seems.

Definition of Proportion



Here $AB : AC = Ab : Ac$ because $BC \parallel bc$.

The definition is "up to congruence". That is, $x : y = u : v$ if segments x, y, u, v are congruent to segments AB, AC, Ab, Ac as shown.

Definition of Proportion

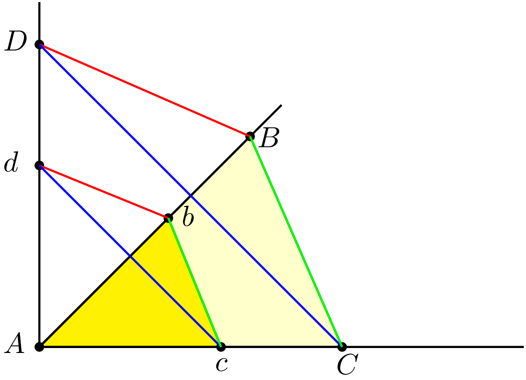
- We defined $AB : CD = ab : cd$ as an 8-argument relation on points.
- It induces a 4-argument relation on (congruence classes of) segments.
- We did not define the ratio $a : b$ or $AB : ab$, so the use of $=$ is just a shorthand, to conform to historical notation.
- Bernays defined $a : b$ to be the acute angle opposite a in a right triangle with legs a and b. Then $=$ is angle congruence, so the parts of the expression are defined. But this is an irrelevant detail. Only the 8-ary relation is ever used.

What is proportion theory?

These are the important theorems (as named by Bernays)

- interchange If $a : b = p : q$, then $a : p = b : q$.
- fundamental theorem If two parallels delineate the segments AC and ac on one side of an angle and AB, Ab on the other side of the angle, then $AB : Ab = AC : Ac$.
- uniqueness of the fourth proportional For each a, b, c, there is exactly one segment x such that $a : b = c : x$ (up to congruence).

The fundamental theorem

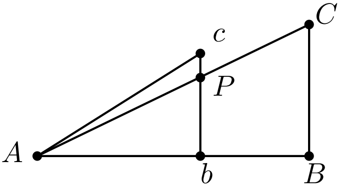


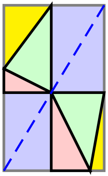
If $BC \parallel bc$ then then $AB : Ab = AC : Ac$.

- It is a consequence of Desargues's theorem.
- Whether it implies Desargues's theorem in some simple way we do not know.
- Bernays proved it by much more elementary means.

A corollary-what we needed to prove $ABC = BCA$

If two right triangles have their hypotenuses and one leg proportional, then their corresponding angles are equal.





$AB : Ab = AC : Ac$ implies angles CAB and cAb are equal.

$$\begin{array}{rcll} AB : Ab & = & AC : Ac & \text{by hypothesis} \\ AB : Ab & = & AC : AP & \text{by the fundamental theorem, since } BC \mid\mid bc \\ Ac & = & AP & \text{by the uniqueness of the fourth proportional} \\ c & = & P & \text{not quite trivial, but it does follow!} \end{array}$$

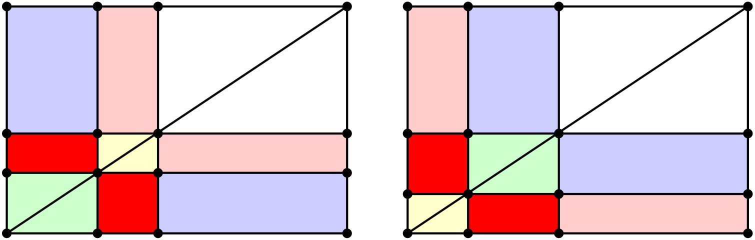
Kupffer and Bernays, final remarks

- The proofs in question are very pretty, but
- they are not quite simple, and
- they are between 60 and 230 years old, and
- we didn't contribute anything new to "early proportion theory".

Since time is limited, we will focus on our own contributions and say no more about Kupffer and Bernays.

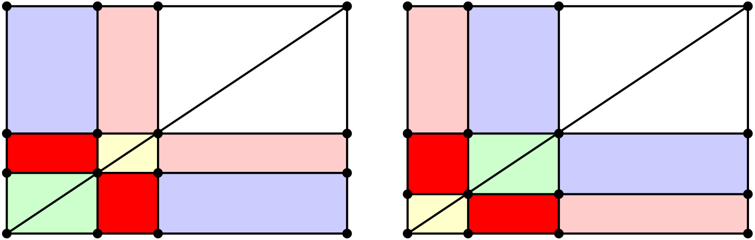
Cutting off equal rectangles from equal rectangles

Cutting off equal rectangles (pink) from equal rectangles (pink and blue) leaves equal rectangles (blue).



Euclid would have justified this by his common notion, “if equals be subtracted from equals, the remainders are equal.”

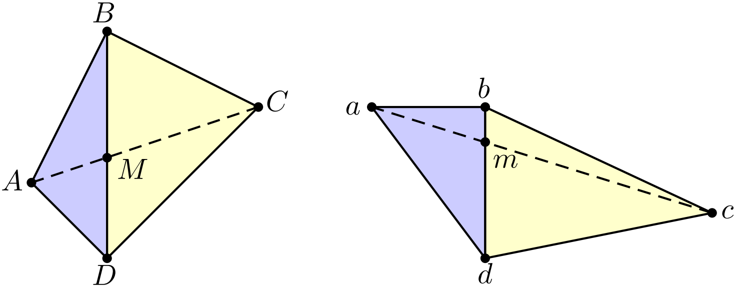
Paste equal (blue) rectangles onto equal (pink) rectangles



- The white rectangles are congruent
- The green rectangles are congruent
- The yellow rectangles are congruent
- The angles of the green and yellow triangles both match those of the white triangles
- Hence all the triangles have the same corresponding angles
- Hence the diagonal lines (defined by their upper 3 points) both pass through the lower left corner too.

Axiom `paste3`

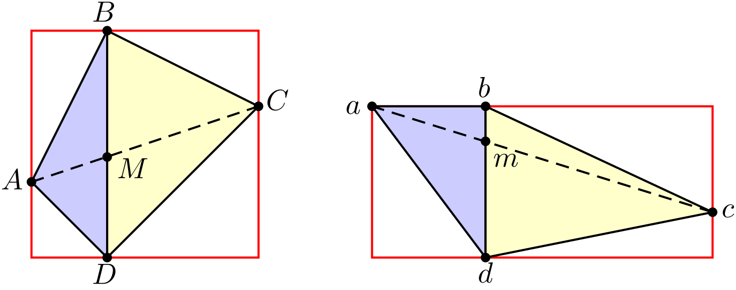
`paste3` is the case of "if equals be added to equals, the wholes are equals", when the equals being added are triangles with a common edge, and the wholes are quadrilaterals.



If the blue triangles are equal, and the yellow triangles are equal, then $ABCD$ and $abcd$ are equal.

Verifying axiom `paste3`

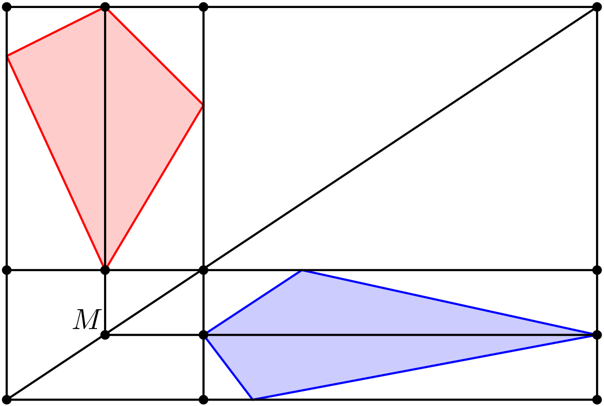
We draw the circumscribing rectangles:



Then the proof reduces to pasting equal rectangles onto equal rectangles.

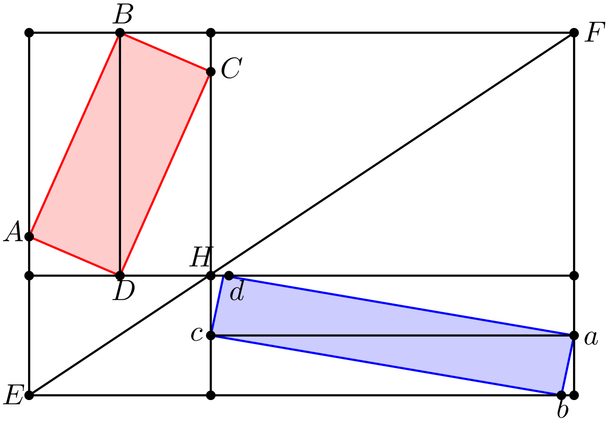
Halves of equals are equal

If equal quadrilaterals are each divided along a diagonal into equal triangles, then all four triangles are equal.



The pink quadrilateral and light blue quadrilateral are equal, and each is divided in two equal triangles. Then M is the center of the lower left rectangle, so the pink and blue triangles are equal.

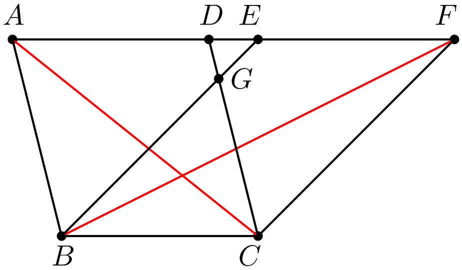
Equal rectangles are also equal quadrilaterals



Triangle ABD is equal to abc as halves of equal rectangles, by definition of equal triangles. One circumscribed rectangle of triangle ABD is half of the circumscribed rectangle of $ABCD$, and one circumscribed rectangle of triangle abc is half of the circumscribed rectangle of $abcd$. Then by `paste3`, quadrilaterals $ABCD$ and $abcd$ are equal.

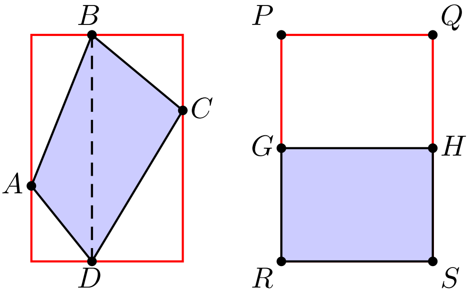
Prop. I.35, proved from the new definitions and theorems

We can use I.37 (triangles with the same base in the same parallels are equal) since that follows immediately from the definition of equal triangles.



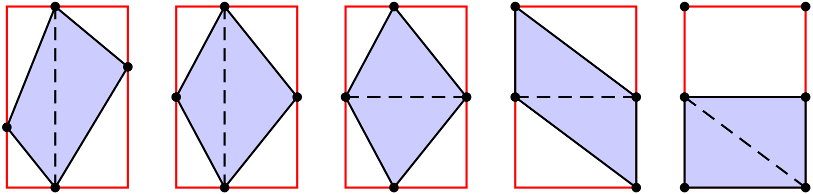
By I.37, triangles ABC and BCF are equal, since they are triangles with the same base and in the same parallels. By I.34, the diagonals AC and BF divide the parallelograms into equal triangles. By `doublesofequals` (a special case of `paste3`), the parallelograms are equal.

Every quadrilateral or triangle is equal to half its circumscribed rectangle



$ABCD$ is equal to half of $PQRS$, namely $GHRS$

Because the blue quadrilaterals are all equal



Those are the main ideas

With those tools, it is possible to complete the proofs of Euclid I.43, I.45, and the rest of the equal-figures axioms.

A Euclidean theory of area

- Prop. I.45 allows to construct a parallelogram with one angle specified that is equal to a given quadrilateral.
- The same method of proof, taken one step further, could have allowed Euclid to construct a parallelogram with one angle *and one side* specified, equal to a given quadrilateral.
- Perhaps Euclid thought that was so obvious that he did not need to spell it out.
- But it could be used to define "equal area." Namely, Two figures have equal area, if they are both equivalent to the same rectangle.
- If one side of the rectangle is regarded as a linear measuring unit, the other side gives the area of the figure measured in square units.
- This way, an area would be a rectangle, not a number or a line.
- But Euclid did not take this step.

Additivity of area

- Using that definition of area, we proved a theorem expressing the additivity of area.
- It only applies to “figures” that are convex quadrilaterals or triangles.
- But it could be generalized.
- We used it to prove a conservative extension theorem:

Theorem. *Any theorem not mentioning “equal figures” that can be proved with the aid of the equal-figures axioms, can also be proved without those axioms, using the defined notions of equal triangle and equal quadrilateral.*

Conclusions

We have given a definition of "equal figures" in the spirit of Euclid, using a diagram similar to the diagram for Prop. I.44. The fundamental properties of this defined notion seem to require (parts of) the theory of proportion. Our work then fell into two parts:

- Using Euclid's methods, as well as the elementary theory of proportions, we proved all the theorems of Book I, and all the equal-figures axioms, using the defined notion of "equal figures."
- We then showed, using theorems of Kupffer and Bernays, that the required theory of proportions can also be developed using the methods of Euclid Book I.

The final result is that it is possible to remove the "equal figures" axioms from the formalization of Euclid Book I that we computer-checked, replacing them with the new definitions and theorems.