Löb's Logic and Lewis' arrow

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Universiteit Utrecht

Overview

Introduction

Arithmetical Interpretation

Uniform Interpolation

Overview

Introduction

Arithmetical Interpretation

Uniform Interpolation

Overview

Introduction

Arithmetical Interpretation

Uniform Interpolation

Overview

Introduction

Arithmetical Interpretation

Uniform Interpolation

The Box and The Arrow

This talk reports on ongoing work with Tadeusz Litak.

The constructive multiverse holds possibilities undreamt of in classical philosophy. We zoom in on one such possibility.

The Lewis arrow does not trivialize. Lewis introduced his arrow $\phi \strictif \psi$ in 1912 out of dissatisfaction with classical implication. However, classically, the arrow and the box are interdefinable:

- $\Box\phi :\leftrightarrow (\top \strictif \phi)$,
- $\phi \strictif \psi :\leftrightarrow \Box(\phi \to \psi)$.

As we shall see, constructively this is not so.

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The Basic System 1

iA has the language of IPC extended with a binary connective $\dashv$. It has the following axioms:

prop	axioms and rules for IPC
N_a	$\vdash \phi \to \psi \quad \Rightarrow \quad \vdash \phi \dashv \psi$
Tr	$((\phi \dashv \psi) \wedge (\psi \dashv \chi)) \to (\phi \dashv \chi)$
K	$((\phi \dashv \psi) \wedge (\phi \dashv \chi)) \to (\phi \dashv (\psi \wedge \chi))$
Di	$((\phi \dashv \chi) \wedge (\psi \dashv \chi)) \to ((\phi \vee \psi) \dashv \chi)$

We take $\Box\phi := (\top \dashv \phi)$.

The Basic System 2

The system iA is our K-analogue. We can derive the usual principles of substitution, etcetera.

We can derive:

$$(\phi \strictif \psi) \leftrightarrow ((\phi \vee \neg \phi) \strictif (\phi \to \psi))$$

but not:

$$\lightning\lightning \qquad (\phi \strictif \psi) \leftrightarrow (\top \strictif (\phi \to \psi)) \qquad \lightning\lightning$$

Principles

$4_\Box$ $\Box\phi \to \Box\Box\phi$

$S_\Box$ $\phi \to \Box\phi$

$L_\Box$ $\Box(\Box\phi \to \phi) \to \Box\phi$

$sL_\Box$ $(\Box\phi \to \phi) \to \phi$

4_a $\phi \strictif \Box\phi$

S_a $(\phi \to \psi) \to (\phi \strictif \psi)$

L_a $(\Box\phi \to \phi) \strictif \phi$

M $\phi \strictif \psi \to (\Box\chi \to \phi) \strictif (\Box\chi \to \psi)$

Box_a $((\chi \wedge \phi) \strictif \psi) \to (\chi \strictif (\phi \to \psi))$

The principle Box_a gives us the reduction to the box. Classical logic does derive Box_a.

Systems

In this talk we will be interested in the following systems.

- $\mathsf{iGL_a} := \mathsf{iA} + \mathsf{L_a}$,
- $\mathsf{iGL} := \mathsf{iGL_a} + \mathsf{Box_a}$,
- $\mathsf{iGL_a} + \mathsf{S}_\Box = \mathsf{iGL_a} + \mathsf{S_a} = \mathsf{iA} + \mathsf{sL}_\Box$,
- $\mathsf{iGL} + \mathsf{S}_\Box = \mathsf{iA} + \mathsf{sL}_\Box + \mathsf{Box_a}$.

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Kripke Semantics 1

A model $\mathcal{K}$ is a quadruple $\langle K, \preceq, \sqsubset, \Vdash_0 \rangle$, where:

- K is non-empty,
- $\preceq$ is a weak partial order on K,
- $k \preceq m \sqsubset n \Rightarrow k \sqsubset n$,
- $\Vdash_0$ relates the nodes in K with propositional variables. It is persistent w.r.t. $\preceq$.

We extend $\Vdash_0$ in the usual way. The clause for $\dashv\!3$ is:

- $k \Vdash \psi \dashv\!3\, \chi$ iff, for all $m \sqsupset k$, we have: $m \Vdash \psi \Rightarrow m \Vdash \chi$.

- $S_{\Box}$ corresponds to: $k \sqsubset m \;\Rightarrow\; k \preceq m$.
- Box_a corresponds to: $k \sqsubset m \preceq n \;\Rightarrow\; k \sqsubset n$.

Overview

Introduction

Arithmetical Interpretation

Uniform Interpolation

HA 1

The modal characterization of the provability logic of HA is one of the two great open problems of provability logic.

The other great open problem is the modal characterization of the provability logic of weak systems like S^1_2 and $I\Delta_0 + \Omega_1$.

We have extra principles like:

- $\Box(\psi \vee \chi) \to \Box(\psi \vee \Box\chi)$,
- $\Box\neg\neg\Box\phi \to \Box\Box\phi$.

HA 2

The most salient interpretation of $\dashv$ is Σ_1-*preservativity*:

- $A \dashv_{T,\Sigma^0_1} B :\leftrightarrow \forall S \in \Sigma^0_1 (\Box_T(S \to A) \to \Box_T(S \to B))$.

Σ_1-preservativity always satisfies Montagna's Principle:

M $\phi \dashv \psi \to (\Box\chi \to \phi) \dashv (\Box\chi \to \psi)$

Σ_1-preservativity is technically a very useful notion in studying the provability logic of HA. It might well be that its logic is simpler than the provability logic of HA. One important feature is that Σ_1-preservativity satisfies Di.

HA^*

Heyting Arithmetic HA has a unique extension HA^* such that, HA-verifiably, $\mathsf{HA}^* = \mathsf{HA} + (A \to \Box_{\mathsf{HA}^*} A)$.

The provability logic of HA^* contains $\mathsf{iGL} + \mathsf{S}_\Box$. However, as was pointed out to me by Mojtaba Mojtahedi, the provability logic of HA^* strictly extends $\mathsf{iGL} + \mathsf{S}_\Box$.

An example:

$$\Box(\Box\bot \to (\neg\phi \to (\psi \vee \chi))) \to \Box(\Box\bot \to ((\neg\phi \to \psi) \vee (\neg\phi \to \chi))).$$

$(\mathsf{HA}^{\mathsf{s}})^*$ 1

HA^{s} is HA axiomatized with only small axioms. Let F_α be an appropriate fast growing hierarchy. An axiom A is small if $\mathsf{F}_{\epsilon_0}(A)$ exists.

We call provability from small axioms *slow provability*.

Sy Friedman, Michael Rathjen and Andreas Weiermann (2013) introduced the notion of slow provability and slow consistency for Peano Arithmetic.

A similar but slightly different idea was studied by Albert Visser (2012) for Elementary Arithmetic.

$(\mathsf{HA}^{\mathsf{s}})^*$ 2

Let $(\mathsf{HA}^{\mathsf{s}})^*$ be the unique theory such that, HA-verifiably,
$(\mathsf{HA}^{\mathsf{s}})^* = \mathsf{HA}^{\mathsf{s}} + (A \to \Box_{(\mathsf{HA}^{\mathsf{s}})^*} A)$.

Zoethout and Visser show that $\mathsf{iGL} + \mathsf{S}_\Box$ is precisely the provability logic of $\widetilde{\mathsf{HA}} := \mathsf{HA} + (A \to \Box_{(\mathsf{HA}^{\mathsf{s}})^*} A) = \mathsf{HA} + (\mathsf{HA}^{\mathsf{s}})^*$.

We may show that $\widetilde{\mathsf{HA}} + \Box_{\widetilde{\mathsf{HA}}} \bot$ strictly extends $\mathsf{HA} + \Box_{\mathsf{HA}} \bot$.

As a corollary we find an alternative proof of the characterization of the provability logic of HA for Σ^0_1-substitutions. This result is originally due to Mohammad Ardeshir & Mojtaba Mojtahedi.

The characterization of the Σ^0_1-provability logic is the most advanced thing we know about the provability logic of HA today.

Overview

Introduction

Arithmetical Interpretation

Uniform Interpolation

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Consider any formula ϕ and any finite set of variables $\widetilde{q}$. Let

$\nu := |\{\psi \in \mathsf{Sub}(\phi) \mid \psi$ is a variable, an implication or a boxed formula$\}|$

We have the following for $\Lambda = \mathsf{iGL_a} + \mathsf{S}_\Box$ or $\Lambda = \mathsf{iGL} + \mathsf{S}_\Box$:

1. There is a formula $\exists \widetilde{q}\, \phi$, the pre-interpolant of ϕ, such that:
   a. $\mathsf{PV}(\exists \widetilde{q}\, \phi) \subseteq \mathsf{PV}(\phi) \setminus \widetilde{q}$
   b. $\mathfrak{c}(\exists \widetilde{q}\, \phi) \leq 2\nu + 2$
   c. For all ψ with $\mathsf{PV}(\psi) \cap \widetilde{q} = \emptyset$, we have:

$$\Lambda \vdash \phi \to \psi \quad \Leftrightarrow \quad \Lambda \vdash \exists \widetilde{q}\, \phi \to \psi.$$

2. There is a formula $\forall \widetilde{q}\, \phi$, the post-interpolant of ϕ, such that:
   a. $\mathsf{PV}(\forall \widetilde{q}\, \phi) \subseteq \mathsf{PV}(\phi) \setminus \widetilde{q}$
   b. $\mathfrak{c}(\forall \widetilde{q}\, \phi) \leq 2\nu + 1$
   c. For all ψ with $\mathsf{PV}(\psi) \cap \widetilde{q} = \emptyset$, we have:

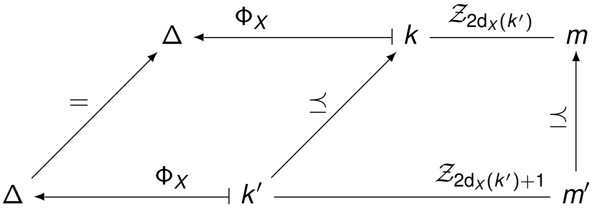
$$\Lambda \vdash \psi \to \phi \quad \Leftrightarrow \quad \Lambda \vdash \psi \to \forall \widetilde{q}\, \phi.$$

A Glimpse of the Proof

Let X be a finite adequate set. We consider prime subsets Δ of X. We construct an model $\widetilde{p},\widetilde{q},\widetilde{r}$-model $\mathcal{N}$ having the right relationships with given $\widetilde{p},\widetilde{q}$-model $\mathcal{K}$ and $\widetilde{p},\widetilde{r}$-model $\mathcal{M}$

A node of $\mathcal{N}$ is a pair $\langle \Delta, m \rangle$ for m in $\mathcal{M}$ for which there is a *witnessing triple* k', k, m'. This means:

$$\Delta = \Delta(k) = \Delta(k'),\ k' \preceq k,\ m' \preceq m,\ k' \mathcal{Z}_{2\mathrm{d}_X(k')+1} m',\ k \mathcal{Z}_{2\mathrm{d}_X(k')} m.$$



Thank You