Uniform reflection in second order arithmetic

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Introduction

- The uniform reflection principle $\mathrm{RFN}(T)$ over a theory T is a schema consisting of sentences

$$\forall x\,(\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x)),$$

where $\varphi(x)$ is a formula with at most the displayed free variable.

$\ulcorner \varphi(\dot{x}) \urcorner$ denotes $\mathrm{sub}(\overline{\ulcorner \varphi \urcorner}, \overline{\ulcorner x \urcorner}, \mathrm{num}(x))$.

- The schema $\mathrm{TI}(\varepsilon_0)$ of transfinite induction up to ε_0 consists of formulas

$$\forall x\,(\forall y \prec x\,\varphi(y) \to \varphi(x)) \to \forall x\,\varphi(x),$$

where $\prec$ defines a primitive recursive well-ordering of order type ε_0.

- In first order arithmetic we have

$$\mathsf{EA} \cup \mathrm{RFN}(\mathsf{EA}) = \mathsf{PA},$$

where EA is Kalmár elementary arithmetic, and

$$\mathsf{PA} \cup \mathrm{RFN}(\mathsf{PA}) = \mathsf{PA} \cup \mathrm{TI}(\varepsilon_0).$$

From Kreisel and Lévy 1968.

- Fine structure for fragments of PA.

For every $n \geq 1$,

$$\mathsf{EA} \cup \mathrm{RFN}_{\Pi_{n+2}}(\mathsf{EA}) = \mathsf{EA} \cup \mathrm{RFN}_{\Sigma_{n+1}}(\mathsf{EA}) = \mathsf{EA} \cup \mathrm{I}\Pi_n.$$

From Leivant 1983.

- Two-sorted language with $x, y, z, \ldots$ for numbers and $X, Y, Z, \ldots$ for sets of numbers.

Signature: $0, 1, +, \cdot, =, <, \in$.

First order terms: $x \mid 0 \mid 1 \mid s + t \mid s \cdot t$.

Second order terms: X.

Formulas: $s = t \mid s < t \mid s \in X \mid \neg, \wedge, \vee, \forall, \exists$.

- A formula is Π^1_n (Σ^1_n) if it is of the form

$$\forall X_1 \, (\exists X_1) \ldots QX_n \, \varphi,$$

where φ is arithmetical, that is, φ does not contain set quantifiers $\forall X$ and $\exists X$.

Full second order arithmetic is:

- PA with induction schema extended to all formulas of second order arithmetic;
- comprehension schema

$$\exists X \, \forall x \, (x \in X \leftrightarrow \varphi(x)).$$

Main subsystems of reverse mathematics: RCA_0 (existence of recursive sets), WKL_0 (existence of paths through 0-1 infinite trees), ACA_0 (existence of Turing jump), ATR_0 (existence of Turing jump iterations along recursive well-orderings), Π^1_1-CA_0 (existence of hyperjump).

Main results

For the rest of the talk, T_0 is a given theory and T is T_0 together with full induction.

We will consider uniform reflection over T_0 and T respectively.

Theorem (Frittaion)

Let $T_0 \supseteq \mathsf{RCA}_0$ be a finitely axiomatizable theory in the language of second order arithmetic. Let T be T_0 together with full induction. Then

$$T_0 \cup \mathrm{RFN}(T_0) = T,$$

and

$$T_0 \cup \mathrm{RFN}(T) = T_0 \cup \mathrm{TI}(\varepsilon_0).$$

The result does not apply to infinite recursively enumerable theories.

For a fine characterization of uniform reflection in second order arithmetic, we need to consider lightface versions of induction and transfinite induction up to ε_0.

Let $(\mathrm{I}\Pi^1_n)^-$ be the restriction of induction to Π^1_n formulas with no set parameters.

Let $(\mathrm{I}\Pi^1_n)^{--}$ be the restriction of induction to Π^1_n formulas with no parameters at all.

Similar definitions apply to $\mathrm{TI}_{\Pi^1_n}(\varepsilon_0)$.

Fragments

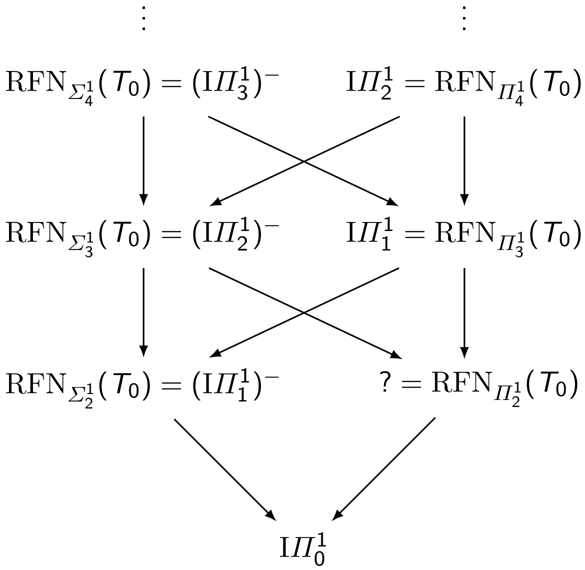
Theorem (Frittaion)

Let T_0 be a Π^1_2 finitely axiomatizable theory containing RCA_0 *and $n \geq 1$. Let T be T_0 plus the schema of full induction. Over T_0,*

$$\mathrm{RFN}_{\Pi^1_{n+2}}(T_0) \;=\; \mathrm{I}\Pi^1_n \;\supsetneq\; (\mathrm{I}\Pi^1_n)^- \;=\; \mathrm{RFN}_{\Sigma^1_{n+1}}(T_0)$$

$$\mathrm{RFN}_{\Pi^1_{n+2}}(T) \;=\; \mathrm{TI}_{\Pi^1_n}(\varepsilon_0) \;\supsetneq\; \mathrm{TI}_{\Pi^1_n}(\varepsilon_0)^- \;=\; \mathrm{RFN}_{\Sigma^1_{n+1}}(T)$$

Over T_0,


$\mathrm{RFN}_{\Sigma^1_4}(T_0) = (\mathrm{I}\Pi^1_3)^-$
$\mathrm{I}\Pi^1_2 = \mathrm{RFN}_{\Pi^1_4}(T_0)$
$\mathrm{RFN}_{\Sigma^1_3}(T_0) = (\mathrm{I}\Pi^1_2)^-$
$\mathrm{I}\Pi^1_1 = \mathrm{RFN}_{\Pi^1_3}(T_0)$
$\mathrm{RFN}_{\Sigma^1_2}(T_0) = (\mathrm{I}\Pi^1_1)^-$
$? = \mathrm{RFN}_{\Pi^1_2}(T_0)$
$\mathrm{I}\Pi^1_0$

Similar diagram for $\mathrm{RFN}(T)$ and $\mathrm{TI}(\varepsilon_0)$.

Under certain hypotheses, the missing arrows denote nonimplications.

Over ACA_0,

- $\mathrm{RFN}_{\Pi^1_n}$ is axiomatized by a Π^1_n sentence, and
- $\mathrm{RFN}_{\Sigma^1_n}$ is axiomatized by an essentially Σ^1_n sentence.

$T \cup \{\varphi\} \nvdash \mathrm{Rfn}_{\neg\varphi}(T) = \mathrm{Pr}_T(\ulcorner\neg\varphi\urcorner) \to \neg\varphi$
(2nd incompleteness).

Uniform reflection is generally stronger than induction and transfinite induction up to ε_0.

Example

Let $T_0 = \mathsf{RCA}_0 \cup \{0^{(n)} \text{ exists} : n \in \omega\}$.

$T_0 \cup \mathrm{RFN}(T_0) \vdash \forall x\,(0^{(x)} \text{ exists})$.

$T_0 \cup \mathrm{TI}(\varepsilon_0)$ does not prove reflection over T_0.

By compactness, there is a model of $T_0 \cup \mathrm{TI}(\varepsilon_0)$ where $\forall x\,(0^{(x)} \text{ exists})$ fails.

Similar examples by using hyperjump.

Proof

(1) From uniform reflection to induction (transfinite induction up to ε_0).

- For every standard n the formula

$$\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1)) \to \varphi(\bar{n})$$

is provable in classical logic.

- For every standard n the formula

$$\forall x\,(\forall y \prec x\,\varphi(y) \to \varphi(x)) \to \forall x \prec \omega_n\,\varphi(x)$$

is provable in RCA (RCA_0 plus full induction).

($\omega_0 = 1$ and $\omega_{n+1} = \omega^{\omega_n}$.)

Formalize (1) and (2) in RCA_0.

(2) From induction (transfinite induction up to ε_0) to uniform reflection.

- Show

$$T = T_0 + \text{full induction} \vdash \mathrm{Pr}_{T_0}(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x),$$

where T_0 is axiomatized by ψ.

Arguing in T, show by induction that every sequent in a finite cut free proof of $\neg\psi, \varphi(\bar{n})$ is true. Use a partial truth definition for, say, formulas of bounded rank. The bound is standard!

In RCA_0 one can prove cut elimination for classical logic.

- Show

$$T_0 \cup \mathrm{TI}(\varepsilon_0) \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x),$$

where T_0 is axiomatized by ψ, and $T = T_0 +$ full induction.

Show by transfinite induction on ε_0 that every sequent in a cut free ω-proof of $\neg\psi, \varphi(\bar{n})$ is true.

In RCA_0 one can prove:

- if $T \vdash \varphi$ then $\vert\!\frac{\omega \cdot 2}{n}\, \neg\psi, \varphi$, for some $n < \omega$;
- if $\vert\!\frac{\alpha}{n}\, \Gamma$ with $n < \omega$, then $\vert\!\frac{\omega_n(\alpha)}{0}\, \Gamma$,

where $\omega_0(\alpha) = \alpha$ and $\omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}$.

Proof for fragments

(1) From uniform reflection to induction (transfinite induction up to ε_0). Count quantifiers!

For instance, if $\varphi(x) \in \Pi^1_n$ has only number parameters (free number variables other than x), then

$$\forall x\,(\forall y \prec x\,\varphi(y) \to \varphi(x)) \to \forall x \prec \omega_z\,\varphi(x)$$

is Σ^1_{n+1} within RCA_0 (by simple quantifier manipulations).

This shows $T_0 \cup \mathrm{RFN}_{\Sigma^1_{n+1}}(T) \vdash \mathrm{TI}_{\Pi^1_n}(\varepsilon_0)^-$,

where T is T_0 plus full induction.

(2) From induction (transfinite induction up to ε_0) to uniform reflection. Count quantifiers and tweak proof by induction (transfinite induction up to ε_0) !

For instance,

$T_0 \cup \mathrm{I}\Pi^1_1 \vdash \mathrm{RFN}_{\Pi^1_3}(T_0)$.

Recall that T_0 is axiomatized by a Π^1_2 sentence $\forall X \, \exists Y \, \psi(X, Y)$ (e.g., ATR_0).

Let $\forall X \, \exists Y \, \varphi(x, X, Y)$ be a Π^1_3 formula with no free variables other than x, where $\varphi(x, X, Y)$ is Π^1_1.

Work in T_0 plus induction for Π^1_1 formulas (with parameters!). Informally. Suppose that $\forall X \, \exists Y \, \varphi(\bar{n}, X, Y)$ is provable in T_0.

We aim to prove that $\forall X \, \exists Y \, \varphi(\bar{n}, X, Y)$ is true. Suppose, for a CONTRADICTION, that there is a set X_0 such that $\forall Y \, \neg\varphi(\bar{n}, X_0, Y)$ is true.

We use the number n and the set X_0 as parameters in a proof by induction of the following fact.

For every sequent Γ in a finite cut free proof of

$$\exists X \, \forall Y \, \neg\psi(X, Y), \forall X \, \exists Y \, \varphi(\bar{n}, X, Y),$$

for any given good interpretation of the free variables, there is a Π^1_1 formula in Γ true under this interpretation.

Conclusion. There must be a true Π^1_1 sentence in $\exists X \, \forall Y \, \neg\psi(X, Y), \forall X \, \exists Y \, \varphi(\bar{n}, X, Y)$. Contradiction.

The only interesting cases are the ones involving the formulas in the end sequent.

- We have an inference of the form

$$\frac{\Gamma, \forall Y\, \neg\psi(U, Y)}{\Gamma, \exists X\, \forall Y\, \neg\psi(X, Y)}$$

Under any interpretation, $\forall Y\, \neg\psi(U, Y)$ is false. In fact, we are assuming $\forall X\, \exists Y\, \psi(X, Y)$.

- We have an inference of the form

$$\frac{\Gamma, \varphi(\bar{n}, U, V)}{\Gamma, \exists Y\, \varphi(\bar{n}, U, Y)}$$

A good interpretation interprets the variable U as the set X_0. Now, $\varphi(\bar{n}, U, V)$ is false under any good interpretation. □

Future

- Study fragments of induction and their parameter free $^{-}$ and $^{--}$ siblings from a model theoretic point of view.
- Study relation with local reflection.
- Study iterations.

References

Emanuele Frittaion. *Uniform reflection in second order arithmetic*. Submitted.

Thanks for your attention!