Sharp actions of groups in the finite Morley rank context

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August 13, 2019

LC'2019, Prague

Outline

1. Overview of sharply 2-transitive groups
2. More structural results, especially for groups of finite Morley rank
3. (Generically) sharply multiply transitive actions

Let G be a group acting on a set X.

Definition.

If for every pair $x, y \in X$ there exists $g \in G$ such that $gx = y$, then we say the action is *transitive*. Moreover, if $g \in G$ is uniquely determined for each pair, then we say the action is *sharply transitive*.

Transitive Actions

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Example.

Rotations of a regular n-gon act sharply transitively on the vertices of the regular n-gon, for $n \geqslant 3$.

Remark.

Every group acts sharply transitively on itself by left multiplication. Thus, sharp transitivity criterion does not bring any restrictions on the group.

Equivalent Actions

We classify actions up to equivalence.

Definition

Let G act on X, and H on Y. If there is a group isomorphism $\alpha : G \to H$ and a bijection $f : X \to Y$ satisfying $f(gx) = \alpha(g)f(x)$ for all $g \in G$ and $x \in X$, then we say that these actions are *equivalent*.

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Example.

$\mathbb{Z}_n$ acting on itself by addition is equivalent to the group of rotations of the regular n-gon acting on the vertices.

Observation.

Every sharply transitive action of a group is equivalent to the left multiplication action of the group on itself. Hence, classification is easy in this case.

Sharp 2-Transitivity

Definition

Let G act on X. If for any distinct $x_1, x_2 \in X$ and distinct $y_1, y_2 \in X$, there exists a (unique) $g \in G$ such that $gx_i = y_i$ for $i = 1, 2$, then we say G acts (sharply) 2-transitively on X.

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Standard Example.

For any field K, the group of affine transformations $\{x \mapsto ax + b \mid a \in K^*, b \in K\}$ acting on K is sharply 2-transitive.

It is equivalent to
$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in K^*, b \in K \right\} \text{ acting on } \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \mid x \in K \right\}.$$
We will write $K^* \ltimes K^+ \curvearrowright K$ for short, to denote this action.

Near-Fields

Remark.

In fact, it is enough to assume that K is a near-field; that is, a skew-field where right distributivity may fail.

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The smallest left near-field which is not a division ring is $\mathbb{F}_9$, where addition is the usual addition and multiplication is defined as:

$$a * b = ab, \text{ if } a \text{ is a square},$$
$$a * b = ab^3, \text{ if } a \text{ is not a square}.$$

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Theorem (Altınel, B., Wagner, 2019)

All infinite near-fields of finite Morley rank and characteristic not 2 are algebraically closed fields.

Splitting

Recall the standard example.

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in K^*, b \in K \right\} \curvearrowright \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \mid x \in K \right\}$$

Observations.

1. The stabilizer of $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in K^* \right\}$.

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2. $B = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in K \right\}$ is an abelian normal subgroup.
3. The groups splits as $G = \mathrm{stab}(x) \ltimes B$. In the study of sharply 2-transitive groups, when we say the group splits, we mean with respect to a stabilizer of a point.

A Natural and Famous Question

Theorem (Karzel, 1971)

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Does every sharply 2-transitive group split?

Some Positive Answers.

- Finite groups (Zassenhaus, 1936)
- Lie groups (Tits, 1952)
- When permutation characteristic is 3 (Kerby, 1972)
- Countable linear groups of $\mathrm{pc} \neq 2$ (Glasner, Gulko, 2014)
- Locally linear groups of odd pc (Glauberman, Mann, Segev, 2015)

A Natural and Famous Question

First Negative Answers.

There are infinite sharply 2-transitive groups that do not split (in permutation characteristics 2 and 0).

Rips–Segev–Tent (2017) and Tent–Ziegler (2016) constructed examples for $\mathrm{pc} = 2$.
Rips–Tent (2019) constructed examples for $\mathrm{pc} = 0$.

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Rips–Tent (2019) constructed examples for $\mathrm{pc} = 0$.

Remark.

None of these examples is of finite Morley rank, so the question is still open in this context.

Involutions in the Standard Example

In the standard example, involutions are of the form

$$\begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix},$$

for every $b \in K$.

Two cases occur.

Either every involution has a fixed point or all involutions are fixed point free, because

$$\begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}$$

iff $b = 2x$.

Let $|X| \geqslant 2$, $G \curvearrowright X$ be sharply 2-transitive, and J be the set of involutions in G.

Observations.

1. $J \neq \emptyset$.

Involutions (General Case)

Let $|X| \geqslant 2$, $G \curvearrowright X$ be sharply 2-transitive, and J be the set of involutions in G.

Observations.

1. $J \neq \emptyset$.
2. J is a single conjugacy class.

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3. Hence, either all involutions have a fixed point, or none has one.

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1. $J \neq \emptyset$.
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3. Hence, either all involutions have a fixed point, or none has one.

Motivated by the standard example, we make the following definition.

Definition

If involutions in G have no fixed points, then we say that the *permutation characteristic* of G is 2. In short, we write $\mathrm{pc}(G) = 2$.

Strongly Real Elements (Standard Example)

Standard Example

In the standard example, strongly real elements are of the form

$$\begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a-b \\ 0 & 1 \end{pmatrix}.$$

Remark.

If the characteristic of the field is $p > 0$, then orders of non-trivial strongly real elements are all equal to p. If the characteristic of the field is 0 then all such elements are torsion-free.

Strongly Real Elements (General Case)

Assume $\mathrm{pc}(G) \neq 2$; that is, assume involutions have fixed points.

Observations.

1. J^2 is a single conjugacy class under the action of G. Hence, elements in J^2 all have the same order.

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2. J^2 is closed under taking powers. Hence, elements of J^2 are either all torsion-free or all of fixed prime order.

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Definition

Depending on the order of elements in J^2, we say that the permutation characteristic of G is 0 or p. We write $\mathrm{pc}(G)$.

Remark.

$\mathrm{pc}(K^* \ltimes K^+) = \mathrm{char}(K)$.

Classification Project for the FMR Context

Problem.

Determine all infinite sharply 2-transitive groups of finite Morley rank up to equivalence. Are they all equivalent to the standard example $K^* \ltimes K^+ \curvearrowright K$ for some algebraically closed field K?

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This problem can be divided into two.

Question 1.

Does every sharply 2-transitive group of finite Morley rank split; that is, can we express $G = \mathrm{stab}(x) \ltimes N$ for some normal subgroup N and $x \in X$?

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Does every sharply 2-transitive group of finite Morley rank split; that is, can we express $G = \mathrm{stab}(x) \ltimes N$ for some normal subgroup N and $x \in X$?

Question 2.

Is every *split* sharply 2-transitive group of finite Morley rank equivalent to the standard example over an algebraically closed field?

What was known?

Infinite sharply 2-transitive groups of finite Morley rank

pc	Q1	Q2
0	?	Cherlin et al, 1991
2	?	?
3	Kerby, 1974	?
$p \geqslant 5$	?	?

Known positive answers before we started working on the problem. There were partial structural results also, which are not shown in the table.
1991 reference is Cherlin, Grundhöfer, Nesin, Völklein.

What did we do?

Infinite sharply 2-transitive groups of finite Morley rank

pc	Q1	Q2
0	?	Cherlin et al, 1991
2	$+$	?
3	Kerby, 1974	$+$
$p \geqslant 5$	?	$+$

$+$: Positive answer given by Altınel, B., Wagner in 2019.

Groups of finite Morley rank

Fact.

In an infinite group of finite Morley rank, all Sylow 2-subgroups are conjugate. If S is one of them, then one of the following holds:

1. $S = 1$,
2. S° is a non-trivial nilpotent group of bounded exponent,
3. S° is a non-trivial divisible abelian group,
4. S° is a product of two such subgroups.

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Depending on the Sylow 2-subgroup structure, G is said to be of degenerate, even, odd, or mixed type, respectively.

Big Fact.

An infinite simple group of finite Morley rank and even type is an algebraic group over an algebraically closed field.

Back to Sharp 2-Transitivity

Proposition

Let $G \curvearrowright X$ be an infinite connected sharply 2-transitive group of finite Morley rank.
(a) If $\mathrm{pc}(G) = 2$, then G is of even type.
(b) If $\mathrm{pc}(G) \neq 2$, then G is of odd type.

Proof.

(a) Through some rank computations, we know that $Y = \bigcup_{x \in X} \mathrm{stab}(x)$ is a generic subset of G. To get a contradiction, assume $S^\circ = T$ is divisible abelian, then $D = \bigcup_{g \in G} C_G(T)^g$ is generic in G, by a result of Cherlin. Moreover, Y and D are disjoint. This contradicts with the connectedness of G.
(b) Distinct involutions do not commute in such a group. $\square$

Socle

Let $G \curvearrowright X$ be a sharply 2-transitive group of finite Morley rank.

Definition

The subgroup of a group generated by minimal normal subgroups is called the *socle* of the group. In the finite Morley rank context, the socle is defined to be generated by infinite definable normal subgroups that are minimal with respect to these properties.

Fact.

An infinite non-abelian group of finite Morley rank contains a non-trivial socle.

In $\mathrm{pc} = 2$, Question 1 is answered.

Theorem (Altınel, B., Wagner, 2019)

Sharply 2-transitive groups of finite Morley rank and of $\mathrm{pc} = 2$ are split.

Proof.

Assume that G has no non-trivial abelian normal subgroups to get a contradiction, and let S be the socle of G. Then one can show that S is an infinite simple group with involutions. Since $\mathrm{pc}(G) = 2$, then S is of even type; hence by the classification, S is an algebraic group over an algebraically closed field. Hence, $\mathrm{stab}(x) \cap S < S$ is an algebraic Frobenius group. This is a contradiction since such groups are split (Hertzig, 1961). $\square$

Permutation Characteristic not 2

When $\mathrm{pc} \neq 2$, Question 2 is answered.

Theorem (Altınel, B., Wagner, 2019)

Split sharply 2-transitive groups of finite Morley rank of $\mathrm{pc} \geqslant 3$ are standard for some algebraically closed field.

Proof is through classifying near-fields of finite Morley rank and characteristic $\geqslant 3$. The case $\mathrm{pc} = 0$ was done by Cherlin et al in 1991.

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When $\mathrm{pc} \neq 2$, Question 2 is answered.

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Proof is through classifying near-fields of finite Morley rank and characteristic $\geqslant 3$. The case $\mathrm{pc} = 0$ was done by Cherlin et al in 1991.
Full conjecture is solved for $\mathrm{pc} = 3$, by using Kerby's result from 1974.

Corollary.

Sharply 2-transitive groups of finite Morley rank of $\mathrm{pc} = 3$ are equivalent to the standard example for some algebraically closed field.

Definition

Let G act on X. If for every pairwise distinct $x_1, \ldots, x_n \in X$ and pairwise distinct $y_1, \ldots, y_n \in X$, there exists a (unique) $g \in G$ such that $gx_i = y_i$ for all $i = 1, \ldots, n$, then we say G acts (sharply) n-transitively on X.

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Examples

For all $n \geqslant 1$, S_n acts sharply n-transitively (also sharply $(n-1)$-transitively) on $\{1, \dots, n\}$. For all $n \geqslant 3$, A_n acts sharply $(n-2)$-transitively on $\{1, \dots, n\}$.

Sharply 3-Transitive Groups

Theorem (Zassenhaus, 1936)

If G is a finite group acting sharply 3-transitively on a set X, then (G, X) is essentially equivalent to $(\mathrm{PGL}_2(K), \mathcal{P}_1(K))$, for some near-field K.

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Theorem (Nesin, 1990)

If G is a group of finite Morley rank acting definably and sharply 3-transitively on a set X, then $(G, X) \cong (\mathrm{PGL}_2(K), \mathcal{P}_1(K))$, for some algebraically closed field K.

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Katrin Tent constructed sharply 3-transitive groups which are not standard in 2016.

Theorem (Jordan, 1872)

Complete list of finite sharply n-transitive groups.
For $n = 4$; S_4, S_5, A_6, M_{11}.
For $n = 5$; S_5, S_6, A_7, M_{12}.
For $n \geqslant 6$; S_n, S_{n+1}, A_{n+2}.

Classification Theorems for $n \geqslant 4$

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Theorem (Tits, 1952, and Hall, 1954)

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There is no infinite group with a sharp n-transitive action, for $n \geqslant 4$.

The story does not end here!

An Alternative Definition for n-transitivity

Observation

Let G be a group acting on a set X and $n \geqslant 2$. Then G acts n-transitively on X iff G acts transitively on $X^n \setminus E$, where $E = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}$.

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Note that E is 'small', hence $X^n \setminus E$ is 'large' or 'generic'. Therefore, we can relax the condition on $X^n \setminus E$ while keeping it large, and obtain new and natural examples.

Definition

Assume that G acts on X. If G is (sharply) transitive on a generic subset of X, then we say G acts generically (sharply) transitively on X.

Generic Transitivity

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Example

Let K be a field, then K^* acts generically sharply transitively on K^+, but not sharply transitively.

Definition

Similarly, if the induced action of G on X^n is generically sharply transitive, then we say G acts generically sharply n-transitively on X.

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- $\mathrm{AGL}_n(K)$ on K^n is generically sharply $(n+1)$-transitive.
- $\mathrm{PGL}_{n+1}(K)$ on $\mathcal{P}_n(K)$ is generically sharply $(n+2)$-transitive.

Some Classification Results

Theorem (B., Borovik, 2018)

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically sharply n-transitively, where $n = \mathrm{rk}(V)$. If V is not a 2-group, then $(G, V) \cong (\mathrm{GL}_n(K), K^n)$.

Small cases $n = 2, 3$ follow from work of Deloro and Borovik–Deloro, respectively.

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Small cases $n = 2, 3$ follow from work of Deloro and Borovik–Deloro, respectively.

Open Question (Borovik, Cherlin, 2008)

Let G be a connected group acting on a set X definably, faithfully and generically sharply $(n+2)$-transitively, where $n = \mathrm{rk}(X)$. Is it true that $(G, X) \cong (\mathrm{PGL}_{n+1}(K), \mathcal{P}_n(K))$, for some algebraically closed field K?

Altınel and Wiscons solved this problem for $n = 2$ and gave a partial result for $n \geqslant 3$.

Thanks

Děkuji!