

# Notions of difference closures of difference fields

Zoé Chatzidakis  
CNRS (UMR 8553) - ENS Paris

Logic Colloquium 19  
Prague, August 11-16, 2019

# Introduction

Two examples of fields with operators: *differential fields* and *difference fields*.

(All fields are commutative)

A differential field is a field  $K$  with an operator  $D$  which is additive and satisfies Cauchy's law:  $D(ab) = aDb + bDa$ .

A difference field is a field  $K$  with a distinguished endomorphism  $\sigma$ . Usually, one imposes to  $\sigma$  to be an automorphism.

The algebra of these fields was developed in parallel by Ritt in the 1930's.

Examples: The field  $\mathbb{C}(t)$ , with the usual derivation  $d/dt$ .

The field  $\mathbb{C}(t)$ , with  $\sigma$  defined as the identity on  $\mathbb{C}$  and sending  $t$  to  $t + 1$  (and so  $f(t)$  to  $f(t + 1)$ ).

## Differentially closed fields (of characteristic 0)

The model companion of the theory of differential fields of characteristic 0,  $\text{DCF}_0$ , was first described by A. Robinson, then later studied by L. Blum. They prove to be very interesting model-theoretically: the theory is complete, eliminates quantifiers, eliminates imaginaries, and moreover is  $\omega$ -stable. The model companion  $\text{DCF}_{0,n}$  of fields of characteristic 0 with  $n$  commuting derivations enjoys similar properties.

In positive characteristic  $p > 0$ , the model companion  $\text{DCF}_p$  of the theory of differential fields of characteristic  $p$  also exists, but is more complicated.

## Differential closure

Since  $\text{DCF}_0$  is  $\omega$ -stable, results of Shelah imply that if  $K$  is a differential field of characteristic 0, then the theory  $\text{DCF}_0$  has a prime model over  $K$ , and which is unique up to  $K$ -isomorphism. This prime model is called the *differential closure* of  $K$ . While its existence and uniqueness are fairly easy to show algebraically when  $K$  is countable, model-theoretic arguments are needed in the uncountable case.

## Difference closed

The model companion of the theory of difference fields exists, ACFA. Its completions are given by describing the isomorphism type of the difference subfield which is the algebraic closure of the prime field, i.e.,  $\bar{\mathbb{Q}}$  or  $\bar{\mathbb{F}}_p$ . The theory ACFA does not eliminate quantifiers, however definable sets are well understood. ACFA eliminates imaginaries. Its completions are not stable, but is supersimple. This means that it has a good notion of independence, and possesses a rank with good properties. The field structure of a model is algebraically closed. But the fixed subfield of a model  $\mathcal{U}$ , i.e.,  $\{a \in \mathcal{U} \mid \sigma(a) = a\}$ , is not algebraically closed, however it is pseudo-finite.

## Difference closure?

In analogy with the differential case, one can call difference fields which are models of ACFA *difference closed*. Then the natural question is:

Do difference fields have a difference closure, and it is unique (up to isomorphism)?

In other words: does the theory ACFA admit prime models over difference fields?

The question has an obvious negative answer, if the underlying field is not algebraically closed: for instance, there are  $2^{\aleph_0}$  incompatible ways of defining an automorphism of  $\bar{\mathbb{Q}}$ , so there can be no prime model of ACFA over  $\mathbb{Q}$ .

So the first condition we need to impose is that the difference field  $K$  be algebraically closed (as a field).

## Second obstacle

As we saw before, the fixed subfield  $F$  of a difference-closed field  $\mathcal{U}$  is pseudo-finite, and its theory has the independence property. It is therefore not surprising that:

if the fixed subfield of  $K$  is not pseudo-finite, then  $K$  does not have a difference-closure.

Indeed, a difference-closed field  $\mathcal{U}$  containing  $K$  will contain an element  $a \notin K$  with  $\sigma(a) = a$ . We'll do the case where  $K$  is algebraically closed, so that  $a$  is transcendental over  $K$ . But there are  $2^{\aleph_0}$  non-isomorphic over  $K$  difference fields  $(K(a)^{alg}, \sigma)$  containing  $K(a)$  and with fixed field pseudo-finite.

## New question

So, in order not to run into these trivial obstacles, we need to make further assumptions:

*We assume that the difference field  $K$  is algebraically closed and that its fixed field  $F$  is pseudo-finite.*

Is it enough to guarantee that the field  $K$  has a difference-closure?

NO ... but the examples are harder.



## Some comments about the examples

So, one needs to find formulas over  $K$  which contain no isolated types.

In characteristic 0, there is a particular example, using the  $j$ -function (a modular function); there are more examples in the same vein, but no infinite family is known.

In positive characteristic  $p > 0$ , there are many families of examples.

However these examples do not answer the question of which difference fields admit a difference closure? Are there some which are not already difference closed?

## Other notions of prime models

A model  $M$  of a theory  $T$  is  $\aleph_\varepsilon$ -saturated if whenever  $A \subset M$  is finite, then every strong type over  $A$  is realised in  $M$ . If  $T = T^{eq}$ , this means that every type over  $\text{acl}(A)$  is realised in  $M$ .

Let  $\kappa$  be an infinite cardinal or  $\aleph_\varepsilon$ ,  $B \subset M$ . One says that  $M$  is  $\kappa$ -prime over  $B$  if  $M$  is  $\kappa$ -saturated, and  $B$ -embeds (elementarily) into every  $\kappa$ -saturated model of  $T$  containing  $B$ .

Shelah showed that if  $T$  is superstable and complete, and  $\kappa$  is either  $> |T|$  or  $\aleph_\varepsilon$  if  $|T| = \aleph_0$ , then  $\kappa$ -prime models exist and are unique up to isomorphism.

This solves the problem of isolated types not being dense in a non totally transcendental theory.

These notions have a direct translation into algebraic terms. For instance, with  $\kappa = \aleph_1$ : A model  $\mathcal{U}$  is  $\aleph_1$ -saturated if and only if every countable set of difference equations (with coefficients in  $\mathcal{U}$ ) which has a solution in a difference field extending  $\mathcal{U}$ , already has a solution in  $\mathcal{U}$ .

The notion of  $\kappa$ -prime then corresponds to a natural notion of closure for this property.

# The result

**Theorem.** Let  $K$  be an algebraically closed difference field of characteristic 0,  $\kappa$  an uncountable cardinal or  $\aleph_\varepsilon$ , and assume that the fixed field  $F$  of  $K$  is  $\kappa$ -saturated. Then  $K$  has a  $\kappa$ -difference closure  $\mathcal{U}$ , and it is unique up to isomorphism over  $K$ .

## Comments

Does not work in characteristic  $p > 0$ . In fact, I am pretty sure that  $\kappa$ -difference closure only exist when  $K$  is already  $\kappa$ -difference closed.

Why does it work in characteristic 0?

## Some ingredients of the proof

(1) The first thing we notice is the following:

Let  $K$  be a difference field, with fixed field  $F$  pseudo-finite and  $\kappa$ -saturated. Then there is a  $\kappa$ -saturated model  $\mathcal{U}$  (of ACFA) containing  $K$  and with fixed field  $F$ .

The proof then follows the usual strategy, and to do that, we need to take a closer look at 1-types:

(2) The generic 1-type: it says that the element does not satisfy any non-trivial difference equation. This type is stationary. In algebraic terms, it is called a transformally transcendental element, and gives rise to a notion of transformal transcendence basis. No problem: a  $\kappa$ -saturated model must have a transformal transcendence basis of cardinality  $\geq \kappa$ , and if needed, we realise the type.

(3) The non-generic types have finite rank and are analysable in terms of types of rank 1. What makes this work in characteristic 0, is that types of rank 1 are either stable stably embedded (and locally modular); or almost internal to the fixed field.

## $\kappa$ -prime models

Under the assumptions of the theorem, show that  $\kappa$ -prime models exist, and are characterized in the usual fashion: every 1-type over  $K$  realised in the model is  $\kappa$ -isolated (it is implied by its restriction to some subset of size  $< \kappa$ ); and every sequence of  $K$ -indiscernibles has length  $\leq \kappa$ . Then show that any two  $\kappa$ -saturated models satisfying these two properties are isomorphic over  $K$ .

The only slightly delicate thing to take care of, are the types internal to  $F$  which are realised in  $\mathcal{U}$ . They are not stable.