Bounds on strong unicity for Chebyshev approximation with bounded coefficients

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Proof mining:

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- tools used: primarily proof interpretations (modified realizability, negative translation, functional interpretation)

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Chebyshev approximation

We have the following classical Chebyshev approximation result.

Theorem (de la Vallée Poussin, Young – 1900s)

For every $n \in \mathbb{N}$ and every continuous $f : [0,1] \to \mathbb{R}$ there is an unique $p \in P_n$ (the set of real polynomials of degree at most n) such that

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Kohlenbach extracted in 1990 a *modulus of uniqueness* – a function Ψ with the property that if p_1 and p_2 are such that $\|f - p_1\|$, $\|f - p_2\| \leq \min + \Psi(\delta)$, then $\|p_1 - p_2\| \leq \delta$.

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He did this by analyzing the uniqueness proof and obtaining an approximate version of it. Let us see how the original proof flows.

A sketch of de la Vallée Poussin's proof

Take p_1 and p_2 that attain the minimum distance E. Then also $\frac{p_1+p_2}{2}$ attains the minimum and we denote it by p.

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$$(p-f)(x_i) = (-1)^{i+j}E.$$

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Let $i \in \{1,\ldots,n+1\}$ and assume wlog that $i+j$ is even. Then $(p-f)(x_i) = E$, so

$$\frac{p_1(x_i)-f(x_i)}{2} + \frac{p_2(x_i)-f(x_i)}{2} = E.$$

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Since $\|p_1 - f\| = E$, $p_1(x_i) - f(x_i) \leq E$. Similarly, $p_2(x_i) - f(x_i) \leq E$. By the above, we have that both are actually equal to E and so $p_1(x_i) = p_2(x_i)$.

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Approximating the proof

Let us now see how one approximates the proof on the previous slide. First, for trivial reasons, the polynomials can be assumed to be in the closed ball Z of radius $\frac{5}{2}\|f\|$ (which is compact, as it lies inside the finite dimensional space P_n).

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1. for all p_1, $p_2 \in Z$ and all $\varepsilon > 0$, if $\|f - p_1\|$, $\|f - p_2\| \leq E + \Phi_1(\varepsilon)$, then $\left\|f - \frac{p_1+p_2}{2}\right\| \leq E + \varepsilon$.

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2. (the "ε-alternation theorem") for all $p \in Z$ and all $\varepsilon > 0$ with $\|f - p\| \leq E + \Phi_2(\varepsilon)$ there is a $j \in \{0, 1\}$ and $x_1 < \ldots < x_{n+1}$ in $[0, 1]$ such that for every $i \in \{1, \ldots, n+1\}$,

$$|(p - f)(x_i) - (-1)^{i+j} E| \leq \varepsilon.$$

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5. for all p_1, $p_2 \in Z$ and all δ, $\beta > 0$, $x_1 < \ldots < x_{n+1}$ in $[0, 1]$ such that for all $i \in \{1, \ldots, n\}$, $x_{i+1} - x_i \geq \beta$ and for all $i \in \{1, \ldots, n+1\}$, $|(p_1 - p_2)(x_i)| \leq \Phi_5(\beta, \delta)$, we have that $\|p_1 - p_2\| \leq \delta$.

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Kohlenbach has extracted moduli Φ_1-Φ_5 and by putting them together he obtained the modulus of uniqueness. This was possible, by the metatheorems of proof mining, because the uniqueness proof could be formalized in WE-PA$^\omega$+WKL+QF-AC^{0,0}.

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The last one is what we are going to focus on.

The result

Theorem (Roulier and Taylor, 1971)

Let $n, m \in \mathbb{N}$ be such that $m \leq n$ and $(k_i)_{i=1}^m \subseteq \mathbb{N}$ be such that $0 < k_1 < \ldots < k_m \leq n$. In addition, let $(a_i)_{i=1}^m$ and $(b_i)_{i=1}^m$ be finite sequences in $\mathbb{R} \cup \{\pm\infty\}$ be such that for all $i \in \{1, \ldots, m\}$, $a_i \leq b_i$, $a_i \neq \infty$ and $b_i \neq -\infty$. If one sets

$$K := \left\{ \sum_{i=0}^{n} c_i X^i \in P_n \;\middle|\; \text{ for all } i \in \{1, \ldots, m\},\; a_i \leq c_{k_i} \leq b_i \right\},$$

then for any continuous $f : [0,1] \to \mathbb{R}$ there is a unique $p \in K$ such that

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The proof resembles the one from before, so we shall focus on the part which is **fundamentally** different.

The approximate form of the new proof

In the ε-alternation step one obtains (among others) an $r \leq n$, a sequence of degrees $n \geq d_1 > d_2 > \ldots > d_{r+1} = 0$ and $x_1 < \ldots < x_{r+1}$ in $[0, 1]$.

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In the last step we deal with the difference $p_1 - p_2$ as before, only we split it as $p_1 - p_2 = Q_1 + Q_2$ where Q_2 has only terms of degrees $d_1, \ldots, d_{r+1}$.

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For Q_2, one must generalize the proof of the original step 5.

Proof of the original step 5

Set $p := p_1 - p_2$. By the classical Lagrangian interpolation formula, we have that:

$$p = \sum_{j=1}^{n+1} \left(\prod_{i \neq j} \frac{X - x_i}{x_j - x_i} \right) \cdot p(x_j).$$

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Since we have, for all $x \in [0, 1]$,

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we get, for all $x \in [0, 1]$,

$$|p(x)| \leq \sum_{j=1}^{n+1} \left| \prod_{i \neq j} \frac{X - x_i}{x_j - x_i} \right| \cdot |p(x_j)| \leq (n+1) \cdot \frac{1}{\beta^n} \cdot \Phi_5(\beta, \delta).$$

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Since we want the right hand side to be smaller or equal to δ, one may take $\Phi_5(\beta, \delta) := \frac{\beta^n}{n+1} \cdot \delta$.

The lemma

Our new step 5 takes the form of the following lemma.

Lemma

Let n, $r \in \mathbb{N}$ with $r \leq n$ and $(d_i)_{i=1}^{r+1} \subseteq \mathbb{N}$ with $n \geq d_1 > d_2 > \ldots > d_{r+1} = 0$. Let β, $\delta > 0$ and $(x_j)_{j=1}^{r+1} \subseteq [0,1]$ such that for all $j \in \{1, \ldots, r\}$, $x_{j+1} - x_j \geq \beta$. Suppose that we have a polynomial

$$p = \sum_{i=1}^{r+1} \eta_i X^{d_i}$$

such that for all $j \in \{1, \ldots, r+1\}$,

$$|p(x_j)| \leq \widetilde{\Phi}_5(\beta, \delta).$$

Then $\|p\| \leq \delta$.

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To obtain $\widetilde{\Phi}_5$, we need to generalize the Lagrangian formula.

Towards the new formula

Using the form of p in the lemma, we get that for all $j \in \{1, \dots, r+1\}$,

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so

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We are thus led to use Vandermonde determinants.

Generalizing Vandermonde

Remember the ordinary Vandermonde determinant:

$$V(y_1,\ldots,y_{r+1}) := \begin{vmatrix} y_1^r & y_1^{r-1} & \cdots & 1 \\ y_2^r & y_2^{r-1} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ y_{r+1}^r & y_{r+1}^{r-1} & \cdots & 1 \end{vmatrix} = \prod_{1\le i<j\le r+1} (y_i - y_j).$$

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Now define the following generalization (where $h_1 > \ldots > h_{r+1}$):

$$V(h_1, \ldots, h_{r+1}; y_1, \ldots, y_{r+1}) := \begin{vmatrix} y_1^{h_1} & y_1^{h_2} & \cdots & y_1^{h_{r+1}} \\ y_2^{h_1} & y_2^{h_2} & \cdots & y_2^{h_{r+1}} \\ \vdots & \vdots & \ddots & \vdots \\ y_{r+1}^{h_1} & y_{r+1}^{h_2} & \cdots & y_{r+1}^{h_{r+1}} \end{vmatrix}.$$

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$$V(y_1, \ldots, y_{r+1}) := \begin{vmatrix} y_1^r & y_1^{r-1} & \cdots & 1 \\ y_2^r & y_2^{r-1} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ y_{r+1}^r & y_{r+1}^{r-1} & \cdots & 1 \end{vmatrix} = \prod_{1 \leq i < j \leq r+1} (y_i - y_j).$$

Now define the following generalization (where $h_1 > \ldots > h_{r+1}$):

$$V(h_1, \ldots, h_{r+1}; y_1, \ldots, y_{r+1}) := \begin{vmatrix} y_1^{h_1} & y_1^{h_2} & \cdots & y_1^{h_{r+1}} \\ y_2^{h_1} & y_2^{h_2} & \cdots & y_2^{h_{r+1}} \\ \vdots & \vdots & \ddots & \vdots \\ y_{r+1}^{h_1} & y_{r+1}^{h_2} & \cdots & y_{r+1}^{h_{r+1}} \end{vmatrix}.$$

Armed with these notations, by expanding the determinant on the previous slide along its first column, we get that

$$p = \sum_{j=1}^{r+1} (-1)^{j-1} \frac{V(d_1, \ldots, d_{r+1}; X, x_1, \ldots, \widehat{x_j}, \ldots, x_{r+1})}{V(d_1, \ldots, d_{r+1}; x_1, \ldots, x_{r+1})} \cdot p(x_j).$$

Young tableaux

We shall need some definitions from algebraic combinatorics to help us in dealing with those determinants.

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- we can move bijectively between strictly decreasing sequences h and partitions λ by the formula $\lambda_i^h := h_i + i - r - 1$
- if $r \in \mathbb{N}$ and λ is a partition of length $r+1$, then a **semistandard Young tableau** of weight λ is a jagged array with $r+1$ rows where for any $i \in \{1, \ldots, r+1\}$, the i'th line has λ_i entries which are elements of the set $\{1, \ldots, r+1\}$, such that the entries on each row are (weakly) increasing and the entries on each column are strictly increasing

1	1	2	7	8
2	3	3		
4	4			
5	6			
6				

Schur functions

Now, if T is such a semistandard Young tableau in which for each $i \in \{1, \dots, r+1\}$, i appears t_i times in T, one denotes by y^T the monomial $y_1^{t_1} \dots y_{r+1}^{t_{r+1}}$. Then the **Schur function** associated to a partition λ is defined by

$$s_\lambda := \sum_T y^T,$$

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The result which is relevant to our ends states that for any r and any strictly decreasing h of length $r+1$,

$$V(h_1, \ldots, h_{r+1}; y_1, \ldots, y_{r+1}) = V(y_1, \ldots, y_{r+1}) \cdot s_{\lambda^h}(y_1, \ldots, y_{r+1}).$$

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A simple proof may be found in:
R. A. Proctor, Equivalence of the combinatorial and the classical definitions of Schur functions. *J. Combin. Theory Ser. A* 51, no. 1, 135–137, 1989.

Back to the formula

The formula for p now becomes

$$p = \sum_{j=1}^{n+1} \left(\prod_{i \neq j} \frac{X - x_i}{x_j - x_i} \right) \cdot p(x_j) \cdot \frac{s_{\lambda^d}(X, x_1, \dots, \widehat{x_j}, \dots, x_{r+1})}{s_{\lambda^d}(x_1, \dots, x_{r+1})}.$$

This formula differs from the Lagrangian one only by the additional Schur factors, so we only need to bound those in order to get $\widetilde{\Phi}_5$.

The upper bound

For any partition λ of length $r+1$, the number of semistandard Young tableaux of weight λ can be shown to be

$$N_\lambda := \prod_{1 \le i < j \le r+1} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

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Proposition

For all n, $r \in \mathbb{N}$ with $r \leq n$, any strictly decreasing h of length $r+1$ and with $h_1 \leq n$, and any $y_1, \ldots, y_{r+1} \in [0,1]$,

$$0 \leq s_{\lambda^h}(y_1, \ldots, y_{r+1}) \leq N_n.$$

The lower bound

First, for all $j \in \{2, \ldots, n\}$, we have that

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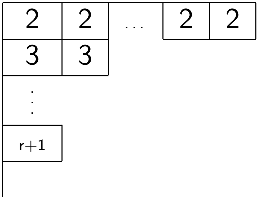

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$$\begin{array}{|c|c|c|c|c|}
\hline
2 & 2 & \ldots & 2 & 2 \\
\hline
3 & 3 & & & \\
\cline{1-2}
\vdots & & & & \\
\cline{1-1}
\text{r+1} & & & & \\
\cline{1-1}
\end{array}$$

we get that

$$\begin{aligned} s_{\lambda^d}(x_1, \ldots, x_{r+1}) &\geq x_2^{\lambda^d_1} \ldots x_{r+1}^{\lambda^d_r} \geq \beta^{\sum_{i=1}^{r} \lambda^d_i} \\ &\geq \beta^{r \cdot \lambda^d_1} = \beta^{r(d_1 - r)} \geq \beta^{r(n-r)} \geq \beta^{\frac{n^2}{4}}. \end{aligned}$$

Wrapping up

We may take then

$$\widetilde{\Phi}_5(\beta, \delta) := \frac{\beta^{n+\frac{n^2}{4}}}{N_n(n+1)} \cdot \delta.$$

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In addition, like with the original Lagrange formula, we may also show the existence of an interpolation polynomial with prescribed degrees, by reversing the above argument (there is a catch, but it is easily taken care of).

The final modulus

Of course, there is much more to the extraction of the modulus. For example, the Schur formula also plays a role in the corresponding ε-alternation result.

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Of course, there is much more to the extraction of the modulus. For example, the Schur formula also plays a role in the corresponding ε-alternation result. In the end, we get the modulus of uniqueness

$$\Psi(\delta) := \frac{\left(\frac{\chi_{\omega,n,M}\left(\frac{L}{2}\right)}{2}\right)^{\frac{n^2}{2}+2n}}{10 \cdot N_n^2 (n+1)(nF_n+1)} \cdot \delta,$$

which depends (in addition to δ) on

- the norm of a polynomial p_0 in K;
- the degree n;
- a lower bound L on E;
- a modulus of uniform continuity ω for f;
- the norm of f.

Some remarks

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- The dependence on the norm of f may be removed at virtually no cost, by a shifting trick.
- The fact that the modulus is linear in δ corresponds to its coefficient being what approximation theorists call a **constant of strong unicity**, the existence of which having been shown before in this setting only nonconstructively.
- One may even remove the dependence on L, though at the expense of linearity.

All this can be found in:

A. Sipoș, Bounds on strong unicity for Chebyshev approximation with bounded coefficients. arXiv:1904.10284 [math.CA], 2019.

Thank you for your attention.