On the Logical Implications of Proof Forms

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Motivation

Proof systems play a crucial role in proof theory, from consistency proofs and proof mining techniques to the characterization of admissible rules.
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Question

Is it possible to prove that some logics do not have a “nice” proof system?
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Question

Is it possible to prove that some logics do not have a “nice” proof system?

This problem has three sides:

- Formalizing nice proof systems;
- considering their corresponding logics;
- finding an invariant, i.e., a property that the logic of a nice proof system enjoys.

Prove almost all logics in a certain given class do not enjoy that property.
Previous work

**Theorem (Iemhoff [5])**

If a *super-intuitionistic* logic has a terminating proof system consisting of *focused rules* and *focused axioms*, it has the uniform interpolation property.
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- A focused rule is a rule with one main formula in its consequence such that the rule respects both the side of this main formula and the occurrence of atoms in it.
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- *Focused axioms* are a modest generalization of the axioms of **LJ**.
- A *focused rule* is a rule with one main formula in its consequence such that the rule respects both the side of this main formula and the occurrence of atoms in it. Example: Conjunction and disjunction rules are focused; implication rules are not.
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- **Focused axioms** are a modest generalization of the axioms of LJ.
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- Nice proof systems are focused proof systems;
- corresponding logics are super-intuitionistic;
- the invariant is uniform interpolation.

Only seven super-intuitionistic logics have uniform interpolation.
Our Contribution

- We will present a second approximation for nice proof systems.
- Our candidate for natural well-behaved sequent-style rules is *semi-analytic* rules (focused rules with no side preserving condition).
- It covers a vast variety of rules: focused rules, implication rules, non-context sharing rules in substructural logics and so many others. We also consider the usual modal rules $K$ and $D$.

Then, we show:
Main Result (informal.)

Theorem (Akbar Tabatabai, J.)

(i) If a sufficiently strong sub-structural logic has a sequent-style proof system only consisting of semi-analytic rules and focused axioms, it has the Craig interpolation property. As a result, many substructural logics and all super-intuitionistic logics, except seven of them, do not have a sequent calculus of the mentioned form.

(ii) If a sufficiently strong sub-structural logic has a terminating sequent-style proof system only consisting of semi-analytic rules and focused axioms, it has the uniform interpolation property. Consequently, \( K4 \) and \( S4 \) do not have a terminating sequent calculus of the mentioned form.

The theorem provides a uniform, proof theoretical and modular method to prove Craig and uniform interpolation.
Craig interpolation

We say a logic $L$ has Craig interpolation property if for any formulas $\phi$ and $\psi$ if $L \vdash \phi \rightarrow \psi$, then there exists formula $\theta$ such that $L \vdash \phi \rightarrow \theta$ and $L \vdash \theta \rightarrow \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$. 
Craig interpolation

We say a logic $L$ has Craig interpolation property if for any formulas $\phi$ and $\psi$ if $L \models \phi \rightarrow \psi$, then there exists formula $\theta$ such that $L \models \phi \rightarrow \theta$ and $L \models \theta \rightarrow \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$.

Uniform interpolation

We say a logic $L$ has the uniform interpolation property if for any formula $\phi$ and any atomic formula $p$, there are two $p$-free formulas, the $p$-pre-interpolant, $\forall p \phi$ and the $p$-post-interpolant $\exists p \phi$, such that $V(\exists p \phi) \subseteq V(\phi)$ and $V(\forall p \phi) \subseteq V(\phi)$ and

(i) $L \models \forall p \phi \rightarrow \phi$,

(ii) For any $p$-free formula $\psi$ if $L \models \psi \rightarrow \phi$ then $L \models \psi \rightarrow \forall p \phi$,

(iii) $L \models \phi \rightarrow \exists p \phi$, and

(iv) For any $p$-free formula $\psi$ if $L \models \phi \rightarrow \psi$ then $L \models \exists p \phi \rightarrow \psi$.

Terminating calculus: there is an order on the sequents...
Basic Sub-structural Logics

$$\phi \Rightarrow \phi \quad \Rightarrow 1 \quad 0 \Rightarrow \quad \Gamma \Rightarrow \top, \Delta \quad \Gamma, \bot \Rightarrow \Delta$$

$$\Gamma \Rightarrow \Delta \quad \Gamma, 1 \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta $$

$$L1 \quad R0$$

$$\Gamma, \phi \Rightarrow \Delta \quad \Gamma \Rightarrow \phi \wedge \psi, \Delta$$

$$\Gamma, \phi \wedge \psi \Rightarrow \Delta \quad R^\wedge$$

$$\Gamma \Rightarrow \phi \wedge \psi, \Delta \quad \Gamma \Rightarrow \psi, \Delta$$

$$L^\lor \quad R^\lor$$

$$\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \phi \lor \psi, \Delta$$

$$\Gamma \Rightarrow \phi \lor \psi, \Delta \quad R^\lor$$

$$\Gamma \Rightarrow \phi, \Delta \quad \Sigma \Rightarrow \psi, \Lambda$$

$$\Gamma \Rightarrow \phi, \Delta \quad R^\ast$$

$$\Gamma, \Sigma \Rightarrow \phi \ast \psi, \Delta, \Lambda$$

$$\Gamma \Rightarrow \phi, \Delta \quad \Sigma, \psi \Rightarrow \Lambda \quad \Gamma \Rightarrow \phi \Rightarrow \psi, \Delta \quad R \Rightarrow$$

$$\Gamma, \Sigma, \phi \rightarrow \psi \Rightarrow \Delta, \Lambda$$

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Basic Sub-structural Logics

- The system consisting of the single-conclusion version of all of the above-mentioned rules is $\text{FL}_e$. 
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In the multi-conclusion case define $\text{CFL}_e$ with the same rules as $\text{FL}_e$, this time in their full multi-conclusion version and add $+$ to the language and the following rules to the system:

\[
\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Sigma, \phi + \psi \Rightarrow \Delta, \Lambda} \quad \frac{\Sigma, \psi \Rightarrow \Lambda}{L^+} \quad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi + \psi, \Delta} \quad \frac{\Gamma, \phi \Rightarrow \Delta}{R^+}
\]
Structural Rules

Weakening rules:

\[
\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \quad Lw \\
\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta} \quad Rw
\]

Contraction rules:

\[
\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \quad Lc \\
\frac{\Gamma \Rightarrow \phi, \phi, \Delta}{\Gamma \Rightarrow \phi, \Delta} \quad Rc
\]
Weakening rules:

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\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \quad Lw \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta} \quad Rw
\]

Contraction rules:

\[
\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \quad Lc \quad \frac{\Gamma \Rightarrow \phi, \phi, \Delta}{\Gamma \Rightarrow \phi, \Delta} \quad Rc
\]

- $\text{FL}_{\text{ew}} = \text{FL}_e + (Lw) + (Rw)$,
- $\text{FL}_{\text{ec}} = \text{FL}_e + (Lc)$,
- $\text{CFL}_{\text{ew}} = \text{CFL}_e + (Lw) + (Rw)$,
- $\text{CFL}_{\text{ec}} = \text{CFL}_e + (Lc) + (Rc)$. 
Semi-analytic rules: Single-conclusion

- **Left semi-analytic rule:**
  \[
  \frac{\langle \langle \Pi_j, \bar{\psi}_js \Rightarrow \bar{\theta}_js \rangle \rangle_j \quad \langle \langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle \rangle_i}{\Pi_1, \ldots, \Pi_m, \Gamma_1, \ldots, \Gamma_n, \phi \Rightarrow \Delta_1, \ldots, \Delta_n}
  \]

  where $\Pi_j$, $\Gamma_i$ and $\Delta_i$'s are meta-multiset variables and

  \[
  \bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi)
  \]
Semi-analytic rules: Single-conclusion

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  \]
  \[
  \Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n
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  where $\Pi_j$, $\Gamma_i$ and $\Delta_i$'s are meta-multiset variables and
  \[
  \bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi)
  \]

- **Right semi-analytic rule:**
  \[
  \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i
  \]
  \[
  \Gamma_1, \cdots, \Gamma_n \Rightarrow \phi
  \]
Semi-analytic rules: Multi-conclusion

- **Left multi-conclusion semi-analytic rule:**

  \[
  \frac{\langle\langle \Gamma_i, \phi \rangle \Rightarrow \psi, \Delta \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}
  \]

- **Right multi-conclusion semi-analytic rule:**

  \[
  \frac{\langle\langle \Gamma_i, \phi \rangle \Rightarrow \psi, \Delta \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n}
  \]
Semi-analytic modal rules

A rule is called *modal semi-analytic* if it has one of the following forms:

\[
\frac{\Gamma \Rightarrow \phi}{\Box \Gamma \Rightarrow \Box \phi} \quad K \quad \frac{\Gamma \Rightarrow \Box \phi}{\Box \Gamma \Rightarrow D}
\]

with the conditions that first, \( \Gamma \) is a meta-multiset variable and secondly whenever the rule \((D)\) is present, the rule \((K)\) must be present, as well.
A generic example of a left semi-analytic rule is the following:

\[ \Gamma, \phi_1, \phi_2 \Rightarrow \psi \quad \Gamma, \theta \Rightarrow \eta \quad \Pi, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta \]

\[ \Gamma, \Pi, \alpha \Rightarrow \Delta \]

where

\[ V(\phi_1, \phi_2, \psi, \theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha) \]
Concrete Examples

Example

The following rules are semi-analytic:

- the usual conjunction, disjunction and implication rules for IPC;
- all the rules in sub-structural logic FL_e, weakening and contraction rules;
- the following rules for exponentials in linear logic:

\[
\begin{array}{c}
\Gamma, !\phi, !\phi \Rightarrow \Delta \\
\hline
\Gamma \Rightarrow \Delta
\end{array}
\]

\[
\begin{array}{c}
\Gamma, !\phi \Rightarrow \Delta \\
\hline
\Gamma, !\phi \Rightarrow \Delta
\end{array}
\]

\[
\begin{array}{c}
\Gamma \Rightarrow \Delta \\
\hline
\Gamma, !\phi \Rightarrow \Delta
\end{array}
\]
Non-examples

Example

- The cut rule; since it does not meet the variable occurrence condition.
- the following rule in the calculus of $\mathbf{KC}$:

\[
\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta}
\]

in which $\Delta$ should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions.
A sequent is called a focused axiom if it has the following form:

1. \((\phi \Rightarrow \phi)\)
2. \((\Rightarrow \bar{\alpha})\)
3. \((\bar{\beta} \Rightarrow)\)
4. \((\Gamma, \bar{\phi} \Rightarrow \Delta)\)
5. \((\Gamma \Rightarrow \bar{\phi}, \Delta)\)

where \(\Gamma\) and \(\Delta\) are meta-multiset variables and in (2) – (5) the variables in any pair of elements in \(\bar{\alpha}\) or \(\bar{\beta}\) or \(\bar{\phi}\) are equal.
Focused axioms

Example

It is easy to see that the axioms given in the preliminaries are examples of focused axioms. Here are some more examples:

\[ \neg 1 \Rightarrow , \Rightarrow \neg 0 \]

\[ \phi, \neg \phi \Rightarrow , \Rightarrow \phi, \neg \phi \]

\[ \Gamma, \neg \top \Rightarrow \Delta , \Gamma \Rightarrow \Delta, \neg \bot \]
Main Result (formal.)

**Theorem**

(i) If $F_{Le} \subseteq L$, and $L$ has a (terminating) single-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then $L$ has Craig (uniform) interpolation.

(ii) If $IPC \subseteq L$ and $L$ has a single-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then $L$ has Craig interpolation.

(iii) If $CFL_{Le} \subseteq L$, and $L$ has a (terminating) multi-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then $L$ has Craig (uniform) interpolation.
Positive Application

As a positive application we have the following:

Corollary

The logics $\text{FL}_e$, $\text{FL}_{ew}$, $\text{CFL}_e$, $\text{CFL}_{ew}$, $\text{CPC}$, and their $\text{K}$ and $\text{KD}$ modal versions have the uniform interpolation property.
As a positive application we have the following:

**Corollary**

The logics $\text{FL}_e$, $\text{FL}_{ew}$, $\text{CFL}_e$, $\text{CFL}_{ew}$, $\text{CPC}$, and their $\text{K}$ and $\text{KD}$ modal versions have the uniform interpolation property.

**Proof.**

The usual sequent calculi for these logics consist of some suitable variants of semi-analytic rules and modal rules.
None of the following logics can have a nice proof system:

- Many substructural logics \((\mathbf{L}_n, \mathbf{L}_\infty, R, BL, \cdots)\);
- Almost all super-intuitionistic logics (except at most seven of them);
- Almost all extensions of \(S4\) (except at most thirty seven of them);
Thank you!

Marchioni E. and Metcalfe G. Craig interpolation for semilinear substructural logics, Mathematical Logic Quartery, Volume 58, Issue 6 November 2012 Pages 468-481.


