On the Logical Implications of Proof Forms

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Motivation

Proof systems play a crucial role in proof theory, from consistency proofs and proof mining techniques to the characterization of admissible rules.

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Question

Is it possible to prove that some logics do not have a "nice" proof system?

This problem has three sides:

- Formalizing nice proof systems;
- considering their corresponding logics;
- finding an invariant, i.e., a property that the logic of a nice proof system enjoys.

Prove almost all logics in a certain given class do not enjoy that property.

Previous work

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- Nice proof systems are focused proof systems;
- corresponding logics are super-intuitionistic;
- the invariant is uniform interpolation.

Only seven super-intuitionistic logics have uniform interpolation.

Our Contribution

- We will present a second approximation for *nice* proof systems.
- Our candidate for natural well-behaved sequent-style rules is *semi-analytic* rules (focused rules with no side preserving condition).
- It covers a vast variety of rules: focused rules, implication rules, non-context sharing rules in substructural logics and so many others. We also consider the usual modal rules K and D.

Then, we show:

Main Result (informal.)

Theorem (Akbar Tabatabai, J.)

(*i*) If a *sufficiently strong sub-structural logic* has a sequent-style proof system only consisting of *semi-analytic* rules and focused axioms, it has the Craig interpolation property. As a result, many substructural logics and all super-intuitionistic logics, except seven of them, do not have a sequent calculus of the mentioned form.

(*ii*) If a *sufficiently strong sub-structural logic* has a **terminating** sequent-style proof system only consisting of semi-analytic rules and focused axioms, it has the uniform interpolation property. Consequently, **K4** and **S4** do not have a terminating sequent calculus of the mentioned form.

The theorem provides a uniform, proof theoretical and modular method to prove Craig and uniform interpolation.

Craig interpolation

We say a logic L has Craig interpolation property if for any formulas ϕ and ψ if $L \vdash \phi \to \psi$, then there exists formula θ such that $L \vdash \phi \to \theta$ and $L \vdash \theta \to \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$.

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Uniform interpolation

We say a logic L has the uniform interpolation property if for any formula ϕ and any atomic formula p, there are two p-free formulas, the p-pre-interpolant, $\forall p\phi$ and the p-post-interpolant $\exists p\phi$, such that $V(\exists p\phi) \subseteq V(\phi)$ and $V(\forall p\phi) \subseteq V(\phi)$ and

(*i*) $L \vdash \forall p\phi \to \phi$,

(*ii*) For any p-free formula ψ if $L \vdash \psi \to \phi$ then $L \vdash \psi \to \forall p\phi$,

(*iii*) $L \vdash \phi \to \exists p\phi$, and

(*iv*) For any p-free formula ψ if $L \vdash \phi \to \psi$ then $L \vdash \exists p\phi \to \psi$.

Terminating calculus: there is an order on the sequents...

Basic Sub-structural Logics

$$\frac{}{\phi \Rightarrow \phi} \qquad \frac{}{\Rightarrow 1} \qquad \frac{}{0 \Rightarrow} \qquad \frac{}{\Gamma \Rightarrow \top, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta}\, L1 \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta}\, R0$$

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta}\, R\wedge$$

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\, R\vee \qquad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\, R\vee$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi * \psi \Rightarrow \Delta}\, L* \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma \Rightarrow \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \phi * \psi, \Delta, \Lambda}\, R*$$

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \to \psi \Rightarrow \Delta, \Lambda}\, L\to \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}\, R\to$$

Basic Sub-structural Logics

- The system consisting of the single-conclusion version of all of the above-mentioned rules is $\mathbf{FL_e}$.

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- The system consisting of the single-conclusion version of all of the above-mentioned rules is $\mathbf{FL_e}$.
- In the multi-conclusion case define $\mathbf{CFL_e}$ with the same rules as $\mathbf{FL_e}$, this time in their full multi-conclusion version and add $+$ to the language and the following rules to the system:

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi + \psi \Rightarrow \Delta, \Lambda} L+ \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi + \psi, \Delta} R+$$

Structural Rules

Weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\, Lw \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}\, Rw$$

Contraction rules:

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\, Lc \qquad \frac{\Gamma \Rightarrow \phi, \phi, \Delta}{\Gamma \Rightarrow \phi, \Delta}\, Rc$$

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- $\mathbf{FL_{ew}} = \mathbf{FL_e} + (Lw) + (Rw)$,
- $\mathbf{FL_{ec}} = \mathbf{FL_e} + (Lc)$,
- $\mathbf{CFL_{ew}} = \mathbf{CFL_e} + (Lw) + (Rw)$,
- $\mathbf{CFL_{ec}} = \mathbf{CFL_e} + (Lc) + (Rc)$.

Semi-analytic rules: Single-conclusion

- *Left semi-analytic rule*:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

where Π_j, Γ_i and Δ_i's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi)$$

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- *Right semi-analytic rule*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi}$$

Semi-analytic rules: Multi-conclusion

- *Left multi-conclusion semi-analytic rule*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

- *Right multi-conclusion semi-analytic rule*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n}$$

Semi-analytic modal rules

A rule is called *modal semi-analytic* if it has one of the following forms:

$$\frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\, K \qquad \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\, D$$

with the conditions that first, Γ is a meta-multiset variable and secondly whenever the rule (D) is present, the rule (K) must be present, as well.

Generic Examples

Example

A generic example of a left semi-analytic rule is the following:

$$\frac{\Gamma, \phi_1, \phi_2 \Rightarrow \psi \qquad \Gamma, \theta \Rightarrow \eta \qquad \Pi, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \Pi, \alpha \Rightarrow \Delta}$$

where

$$V(\phi_1, \phi_2, \psi, \theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

Concrete Examples

Example

The following rules are semi-analytic:

- the usual conjunction, disjunction and implication rules for **IPC**;
- all the rules in sub-structural logic $\mathbf{FL_e}$, weakening and contraction rules;
- the following rules for exponentials in linear logic:

$$\frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

Non-examples

Example

- The cut rule; since it does not meet the variable occurrence condition.
- the following rule in the calculus of **KC**:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

in which Δ should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions.

Focused axioms

Focused axioms

A sequent is called a *focused axiom* if it has the following form:

(1) $(\phi \Rightarrow \phi)$

(2) $(\Rightarrow \bar{\alpha})$

(3) $(\bar{\beta} \Rightarrow)$

(4) $(\Gamma, \bar{\phi} \Rightarrow \Delta)$

(5) $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are meta-multiset variables and in $(2) - (5)$ the variables in any pair of elements in $\bar{\alpha}$ or $\bar{\beta}$ or $\bar{\phi}$ are equal.

Focused axioms

Example

It is easy to see that the axioms given in the preliminaries are examples of focused axioms. Here are some more examples:

$$\neg 1 \Rightarrow \quad , \quad \Rightarrow \neg 0$$

$$\phi, \neg\phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg\phi$$

$$\Gamma, \neg\top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg\bot$$

Main Result (formal.)

Theorem

(i) *If* $\mathbf{FL_e} \subseteq L$*, and* L *has a (terminating) single-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then* L *has Craig (uniform) interpolation.*

(ii) *If* $\mathbf{IPC} \subseteq L$ *and* L *has a single-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then* L *has Craig interpolation.*

(iii) *If* $\mathbf{CFL_e} \subseteq L$*, and* L *has a (terminating) multi-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then* L *has Craig (uniform) interpolation.*

Positive Application

As a positive application we have the following:

Corollary

The logics $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, $\mathbf{CPC}$, *and their* $\mathbf{K}$ *and* $\mathbf{KD}$ *modal versions have the uniform interpolation property.*

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Proof.

The usual sequent calculi for these logics consist of some suitable variants of semi-analytic rules and modal rules. $\square$

Negative Applications

Corollary

None of the following logics can have a **nice** *proof system:*

- *Many substructural logics ($Ł_n$, $Ł_\infty$, R, BL, $\cdots$);*
- *Almost all super-intuitionistic logics (except at most seven of them);*
- *Almost all extensions of* **S4** *(except at most thirty seven of them);*

Thank you!

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