The limit of incompleteness for Weak Arithmetics

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Logic Colloquium 2019

August 11-16, Prague

Gödel's incompleteness theorems

Theorem 1 (Gödel)

Let T be a recursive enumerable (r.e.) extension of **PA***.*

First incompleteness theorem (G1) *If T is ω-consistent, then T is not complete (there is a sentence θ such that $T \nvdash \theta$ and $T \nvdash \neg\theta$).*

Second incompleteness theorem (G2) *If T is consistent, then the consistency of T is not provable in T.*

- We say a consistent r.e. theory T is essentially incomplete if any consistent r.e. extension of T is incomplete.
- **PA** is essentially incomplete.

Outline of the content

- Classifications of different proofs of Gödel's incompleteness theorems
- The limit of applicability of G1
- The limit of applicability of G2

Different proofs of incompleteness

We could classify proofs of Gödel's incompleteness theorems from the following aspects:

- Proof via proof theoretic method;
- Proof via recursion theoretic method;
- Proof via model theoretic method;
- Proof via arithmetization;
- Proof via the fixed point lemma;
- Proof via logical paradox;
- Proof via constructive method;
- Proof only assuming that the base theory is consistent;
- Independent sentences with real mathematical content.

Incompleteness without arithmetization

Grzegorczyk proposed the theory of concatenation (**TC**) with no reference to natural numbers and proved that **TC** is essentially incomplete without arithmetization.

Definition 1 (A. Grzegorczyk)

The theory of concatenation **TC** *has the language* $\{\frown, \alpha, \beta\}$ *and the following axioms:*

- TC1 $\forall x \forall y \forall z (x \frown (y \frown z) = (x \frown y) \frown z)$;
- TC2 $\forall x \forall y \forall u \forall v (x \frown y = u \frown v \to ((x = u \wedge y = v) \vee \exists w ((u = x \frown w \wedge w \frown v = y) \vee (x = u \frown w \wedge w \frown y = v))))$;
- TC3 $\forall x \forall y (\alpha \neq x \frown y)$;
- TC4 $\forall x \forall y (\beta \neq x \frown y)$;
- TC5 $\alpha \neq \beta$.

Incompleteness and logical paradox

Different proofs of incompleteness theorems via paradox:

Gödel Liar Paradox

Boolos, Chaitin, Kikuchi, Vopenka, Kurahashi, Sakai, Tanaka Berry's paradox

Kikuchi, Kurahashi, Priest, Cieśliński and Urbaniak Yablo's Paradox

Kritchman-Raz Unexpected Examination Paradox

Cieśliński Grelling-Nelson's Paradox

Concrete incompleteness for **PA**

Gödel's proof uses meta-mathematical method and Gödel's sentence has no real mathematical content.

A natural question is then: can we find true sentences not provable in PA with real mathematical content?

Paris-Harrington Paris-Harrington principle

Kirby and Paris The Goodstein sequence, The Hercules-Hydra game

Kanamori-McAloon The Kanamori-McAloon principle

Beklemishev The Worm principle

Kirby The flipping principle

Mills The arboreal statement

Pudlák P.Pudlák's Principle

Clote The kiralic and regal principles

Weiermann Variants of Paris-Harrington principle and Goodstein sequence

Concrete incompleteness for Higher-Order Arithmetic

Question 1

Can we find a mathematical theorem expressible in Second-Order Arithmetic but not provable in Second-Order Arithmetic?

Theorem 2

There is a concrete mathematical theorem which is expressible in Second-Order Arithmetic, not provable in Second-Order Arithmetic, not provable in Third-Order Arithmetic, but provable in Fourth-Order Arithmetic.

Reference:

Yong Cheng. Incompleteness for Higher-Order Arithmetic: An example based on Harrington's Principle. Springer series: Springerbrief in Mathematics, 2019.

The notion of interpretation

- An interpretation of a theory T in a theory S is a mapping from formulas of T to formulas of S that maps all axioms of T to sentences provable in S.
- $S \trianglelefteq T$ denotes that S is interpretable in T.
- $S \lhd T$ denotes that S is interpretable in T but T is not interpretable in S (i.e. S is weaker than T w.r.t. interpretation).

Finding the limit of applicability of G1

Question 2 (The big open question)

Exactly how much information of arithmetic is needed for the proof of G1*?*

Definition 2

G1 *holds for r.e. theory T iff for any consistent r.e. theory S, if T is interpretable in S, then S is incomplete.*

Proposition 1

G1 *holds for T iff T is essentially incomplete.*

The system $\mathbf{R}$

Definition 3 (Tarski, Mostowski and Robinson)

Let $\mathbf{R}$ *be the system consisting of schemes* $\mathbf{Ax}1$-$\mathbf{Ax}5$ *with* $L(\mathbf{R}) = \{\mathbf{0}, \overline{n}, +, \cdot, \leq\}$ *where* $m, n \in \mathbb{N}$.

*Ax*1 $\overline{m} + \overline{n} = \overline{m+n}$;

*Ax*2 $\overline{m} \neq \overline{n}$ *if* $m \neq n$;

*Ax*3 $\overline{m} \cdot \overline{n} = \overline{m \cdot n}$;

*Ax*4 $\forall x(x \leq \overline{n} \rightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$;

*Ax*5 $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$;

Summary

$I\Sigma_n$ is Robinson arithmetic $\mathbf{Q}$ plus induction for Σ_n formulas.

In a summary, we have the following picture:

(1) $\mathbf{Q} \lhd I\Sigma_1 \lhd I\Sigma_2 \lhd \cdots \lhd I\Sigma_n \lhd \cdots \lhd \mathbf{PA}$, and G1 holds for them;

(2) Theories $\mathbf{PA}^-, \mathbf{Q}^+, \mathbf{Q}^-, \mathbf{TC}, \mathbf{AS}, \mathbf{S}^1_2$ and $\mathbf{Q}$ are all mutually interpretable and hence G1 holds for them;

(3) $\mathbf{R} \lhd \mathbf{Q} \lhd \mathbf{PA}$ and G1 holds for them.

Theorem 3 (Visser)

Suppose $\mathbf{R} \subseteq A$, where A is finitely axiomatized and consistent. Then there is a finitely axiomatized B such that $\mathbf{R} \subseteq B \subseteq A$ and $B \lhd A$.

G1 holds for many theories weaker than $\mathbf{R}$

Question 3

Could we find a theory S such that G1 *holds for S and $S \lhd \mathbf{R}$?*

Definition 4

$\langle S, T\rangle$ is a recursively inseparable pair if S and T are disjoint r.e. sets, and there is no recursive set $X \subseteq \mathbb{N}$ such that $S \subseteq X$ and $X \cap T = \emptyset$.

Theorem 4

For any recursively inseparable pair $\langle A, B\rangle$, there is a theory $U_{\langle A,B\rangle}$ such that G1 *holds for $U_{\langle A,B\rangle}$ and $U_{\langle A,B\rangle} \lhd \mathbf{R}$.*

Definition 5

Let $\langle A, B\rangle$ be a recursively inseparable pair. The theory $U_{\langle A,B\rangle}$ consists of the following axioms in the language $\{\mathbf{0}, \overline{n}, \mathbf{P}\}$:

(1) $\overline{m} \neq \overline{n}$ *if* $m \neq n$;

(2) $\mathbf{P}(\overline{n})$ *if* $n \in A$;

(3) $\neg\mathbf{P}(\overline{n})$ *if* $n \in B$.

The difficult part is to show that $U_{\langle A,B\rangle}$ does not interpret **R**.

I show this using some tools from JeŘábek's work via model theory.

Reference:

Emil JeŘábek, Recursive functions and existentially closed structures, to appear in Journal of Mathematical Logic.

Yong Cheng, Finding the limit of incompleteness I, Submitted.

A question

Theorem 5 (Visser)

Suppose T is an r.e. theory. Then T is interpretable in $\mathbf{R}$ iff T is locally finitely satisfiable.

Question 4 (Visser)

Would S with $S \trianglelefteq \mathbf{R}$ such that G1 *holds for S shares the universality property of $\mathbf{R}$ that every locally finitely satisfiable theory is interpretable in it.*

The answer for this question is negative.

For any recursively inseparable pair $\langle A, B\rangle$, the theory $U_{\langle A,B\rangle}$ is a counterexample for Visser's Question.

Questions

Define $\mathsf{D} = \{\mathsf{S} : \mathsf{S} \lhd \mathbf{R}$ and G1 holds for $S\}$.

Question 5

(1) *Could we find a minimal theory S w.r.t. interpretation such that* G1 *holds for S?*

(2) *Is* $(\mathsf{D}, \lhd)$ *well founded?*

(3) *Are any two elements of* $(\mathsf{D}, \lhd)$ *comparable?*

Conjecture 1

$(\mathsf{D}, \lhd)$ *is not well founded and has incomparable elements.*

Turing degree verse Interpretability degree

Define $\mathrm{D} = \{\mathrm{S} : \mathrm{S} <_{\mathrm{T}} \mathbf{R} \text{ and } \mathrm{G1} \text{ holds for } S\}$.

Question 6

(1) *Could we find a minimal theory S w.r.t. Turing Reducibility such that* G1 *holds for S?*

(2) *Is* $(\mathrm{D}, <_{\mathrm{T}})$ *well founded?*

(3) *Are any two elements of* $(\mathrm{D}, <_{\mathrm{T}})$ *comparable?*

Theorem 6

For any Turing degree $\mathbf{0} < \mathbf{d} < \mathbf{0}'$*, there is a theory* U *such that* G1 *holds for* U*,* $U <_T \mathbf{R}$ *and* U *has Turing degree* $\mathbf{d}$*.*

Corollary 1

(1) *There is no a minimal theory* S *w.r.t. Turing Reducibility such that* G1 *holds for* S*?*

(2) $(\mathrm{D}, <_{\mathrm{T}})$ *is not well founded.*

(3) $(\mathrm{D}, <_{\mathrm{T}})$ *has incomparable elements.*

The intensionality of G2

- We say that G2 holds for T if the consistency statement of T is not provable in T.
- This definition is vague: what do we mean "the consistency statement of T is not provable in T"?
- G2 is essentially different from G1 due to the intensionality of G2: whether G2 holds for T depends on how we formulate the consistency statement.

Factors affecting G2

"Whether G2 holds for T" depends on the following factors:

(1) the definition of provability predicate;

(2) the choice of an arithmetic formula to express consistency;

(3) the choice of the base proof system;

(4) the choice of numberings;

(5) the choice of a specific formula numerating (representing) the axiom set.

G2 and the definition of provability predicate

- The consistency statement $\mathbf{Con}(T)$ is usually defined as $\neg\mathbf{Pr}_T(\ulcorner 0 \neq 0 \urcorner)$.
- Being a consistency statement is not an absolute concept but a role w.r.t. a choice of the provability predicate.
- G2 holds for any provability predicate which satisfies the Hilbert-Bernays-Löb Derivability Condition **D1-D3**.
- Define the Rosser provability predicate $\mathbf{Pr}_T^R(x)$ as the formula $\exists y(\mathbf{Prf}_T(x,y) \wedge \forall z \leq y \neg\mathbf{Prf}_T(\dot{\neg}(x), z))$.
- G2 fails for Rosser provability predicate: $T \vdash \mathbf{Con}^R(T) \triangleq \neg\mathbf{Pr}_T^R(\ulcorner 0 \neq 0 \urcorner)$.

G2 and the choice of arithmetic formulas to express consistency

(1) $\mathbf{Con}(T) \triangleq \neg\mathbf{Pr}_T(\ulcorner \mathbf{0} \neq \mathbf{0} \urcorner)$.

(2) $\mathbf{Con}^0(T) \triangleq \forall x(\mathbf{Fml}(x) \wedge \mathbf{Pr}_T(x) \rightarrow \neg\mathbf{Pr}_T(\dot{\neg} x))$;

Kurahashi constructed a Rosser provability predicate such that G2 holds for the consistency statement formulated via $\mathbf{Con}^0(T)$, but G2 fails for the consistency statement formulated via $\mathbf{Con}(T)$.

G2 and the choice of numberings

Any injective function γ from a set of $L(\mathbf{PA})$-expressions to $\mathbb{N}$ qualifies as a numbering.

Gödel's numbering is a special kind of numberings under which the Gödel number of the set of axioms of **PA** is recursive.

"Whether G2 holds for T" depends on the choice of numberings.

Grabmayr shows that G2 holds for acceptable numberings; But G2 fails for some non-acceptable numberings.

G2 depends on the numeration of T

- $\alpha(x)$ is a numeration of **PA** if for any n, **PA** $\vdash \alpha(\overline{n})$ iff n is the Gödel number of some sentence in T.
- G2 holds for Σ_1 numerations of **PA**, but fails for some Π_1 numerations of **PA**.
- Feferman constructs a Π_1 numeration $\tau(u)$ of **PA** such that G2 fails under this numeration.

G2 and the choice of base system

- An foundational question about G2 is: how much of information about arithmetic is required for the proof of G2. If the base proof system does not contain enough information about arithmetic, then G2 may fail.
- Dan Willard has constructed examples of c.e. arithmetical theories that couldn't prove the totality of successor function but could prove their own canonical consistency.
- Pakhomov defined a weak set theory $H_{<\omega}$ and showed that it proves its own consistency.

G1 versus G2

- G2 holds for any consistent r.e. theory interpreting **Q**.
- But it is not true that G2 holds for any consistent r.e. interpreting **R**.
- If $S \trianglelefteq T$ and G1 holds for S, then G1 holds for T.
- But it is not true that: if $S \trianglelefteq T$ and G2 holds for S, then G2 holds for T.

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Thanks for your attention!