Feasible incompleteness

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Notation

- We denote by letters $p, q, r, p_1, q_1 \dots$ polynomial functions.
- $\mathcal{T}$ is the class of all consistent arithmetical theories that extend Buss's theory S^1_2 by a set of axioms that is in the complexity class P.
- If φ is a formula with Gödel number n, then $\overline{\varphi}$ denotes a closed term $\overline{n}$
- Moreover, if $\varphi(x)$ is a formula with one free variable x, then $\overline{\varphi(\dot{x})}$ denotes a formalization of the function "$n \mapsto$ Gödel number of the sentence $\varphi(\overline{n})$
- We will denote by the formula $Proof_S(x, y)$ a natural formalization of the relation "x is a S-proof of y"
- A formula $P_T(y)$ is then defined as

$$P_T(y) \equiv_{df} \exists x Proof_T(x, y)$$

Notation

- With the help of the formula $Proof_S(x, y)$, we define a formula $Pr_S(z, y)$ as a natural formalization of the relation:
  "There exists a S-proof of y of the length shorter than z"
- Consistency of a given theory T, Con_T, is then defined as the sentence

$$Con_T \equiv_{df} \neg P_T(\overline{0 = S(0)})$$

- Finite consistency, $Con_T(x)$, is defined with the help of the formula $Pr_T(x, y)$ in the following way

$$Con_T(x) \equiv_{df} \neg Pr_T(x, \overline{0 = S(0)})$$

Finite versions of Gödel's incompleteness theorems

Conjecture (CON, P. Pudlak 1986)

Let $S,\ T \in \mathcal{T}$ be theories such that

$$S + Con_S = T$$

Then the length of S-proofs of $Con_T(\overline{n})$ cannot be bounded by any polynomial function in n.

- Connection to open problems in computational complexity theory

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Theorem

Assume conjecture CON, then $NEXP \neq coNEXP$

Finite versions of Gödel's incompleteness theorems

- We want to state a finite version of Gödel's first incompleteness theorem.
- With the help of Diagonal lemma, define a formula $\varphi(x)$ such that for $S \in \mathcal{T}$

$$S \vdash \varphi(x) \equiv \neg Pr_S(x, \overline{P_S(\overline{\varphi(\dot{x})})})$$

Lemma

Let $S \in \mathcal{T}$ be a theory and let $\varphi(x)$ be as above. Then there exist polynomial functions q_1 and $q_2(n) = O(n)$ such that the following holds

$$S \vdash \forall x (Con_{S+Con_S}(\overline{q_1(\dot{x})}) \to \varphi(x))$$

$$S \vdash \forall x (\varphi(\overline{q_2(\dot{x})}) \to Con_{S+Con_S}(x))$$

Finite versions of Gödel's incompleteness theorems

- Thus, we obtain for $S \in \mathcal{T}$

$$S \vdash^{p(n)} \varphi(\overline{n}) \Leftrightarrow S \vdash^{p(n)} Con_{S+Con_S}(\overline{n})$$

Conjecture (Finite version of Gödel's first incompleteness theorem, F1GT)

Let $S \in \mathcal{T}$ and let $\varphi(x)$ be a formula such that

$$S \vdash \varphi(x) \equiv \neg Pr_S(x, \overline{P_S(\overline{\varphi(\dot{x})})})$$

Then the length of S-proofs of the sentence $\varphi(\overline{n})$ is not bounded by any polynomial function in n.

- The conjecture F1GT is equivalent to the conjecture CON

Π_2 independent sentence

Theorem (P. Pudlak, 1986)

Let $T \in \mathcal{T}$. Then there exists a polynomial function p such that

$$T \vdash \forall x Pr_T(p(x), \overline{Con_T(\dot{x})})$$

- Let $S \in \mathcal{T}$ and, moreover, let S be Σ_1-sound theory. Then

$$\mathbb{N} \models \forall y P_S(\overline{\varphi(\dot{y})})$$

$$S \nvdash \forall y P_S(\overline{\varphi(\dot{y})})$$

- Moreover, for some polynomial function p_3

$$S \vdash P_S(\overline{Con_{S+Con_S}(\overline{p_3(\dot{x})})}) \rightarrow P_S(\overline{\varphi(\dot{x})})$$

Π_2 independent sentence

Theorem

Let S be a theory such that $S \in \mathcal{T}$ and, moreover, let S be Σ_1-sound. Then

$$\mathbb{N} \models \forall x P_S(\overline{Cons_{S+Cons_S}(\dot{x})})$$

$$S \nvdash \forall x P_S(\overline{Cons_{S+Cons_S}(\dot{x})})$$

It is interesting to ask what causes the independence of the sentence from the Theorem above

- The sentence

$$\forall x P_S(\overline{Cons_{S+Cons_S}(\dot{x})})$$

  is Π_2 sentence

$$\forall x \exists y Proof_S(y, \overline{Cons_{S+Cons_S}(\dot{x})})$$

Π_2 independent sentence

- Π_2 sentence can be interpreted as a total function defined on N. Thus, there may be a possible analogy with fast growing functions.
- The unprovability of

$$\forall x \exists y Proof_S(y, \overline{Con_{S+Con_S}(\dot{x})})$$

  can be caused by the lengths of proofs of the formula $Con_{S+Con_S}(x)$. The function in may be growing exponentially (this is in agreement with the conjecture CON)

Feasible finite independence

- We would like to find a formula $\varphi(n)$ such that the both formulas $\neg Pr_T(\overline{n}, \overline{\varphi(\overline{n})})$ and $\neg Pr_T(\overline{n}, \overline{\neg\varphi(\overline{n})})$ have in $T \in \mathcal{T}$ proofs of the polynomial length in n
- Let $\psi(x)$ be Rosser's formula without the universal quantifier, that is, define $\psi(x)$ in the following way:

$$T \vdash \psi(x) \equiv (Pr_T(x, \overline{\psi(\dot{x})}) \to \exists v \leq x Pr_T(v, \overline{\neg\psi(\dot{x})}))$$

Theorem

Let $T \in \mathcal{T}$ be a theory and $\psi(x)$ be as above. Then there exists a polynomial function p such that for all $n \in N$

$$T \vdash^{p(n)} \neg Pr_T(\overline{n}, \overline{\psi(\overline{n})}) \tag{1}$$

and

$$T \vdash^{p(n)} \neg Pr_T(\overline{n}, \overline{\neg\psi(\overline{n})}) \tag{2}$$

Finite version of Löb theorem

- After Kurt Gödel proved his famous theorems, L. Henkin asked an interesting question of what is equivalent in a sufficiently strong theory T a sentence ψ such that

$$T \vdash \psi \equiv P_T(\overline{\psi})$$

- The answer was found by M. Löb. If ψ is a sentence such that

$$T \vdash P_T(\overline{\psi}) \rightarrow \psi$$

then already

$$T \vdash \psi$$

Finite version of Löb theorem

- In the similar way we can ask whether on the basis of finite version of Gödel's first incompleteness theorem, the conjecture F1GT, what implies

$$T \vdash^{p(n)} Pr_T(\overline{n}, \overline{P_T(\overline{\psi(\overline{n})})}) \rightarrow \psi(\overline{n}) \qquad (1)$$

  for some formula $\psi(x)$ and a polynomial function p.
- Here, the concept of provability is replaced by the concept of polynomial provability or, philosophically speaking, "feasible provability".

Finite version of Löb theorem

Conjecture (Finite version of Löb's theorem, FL)

Let $T \in \mathcal{T}$ be a theory and $\psi(x)$ a formula. If there exists polynomial function p such that for every n

$$T \vdash^{p(n)} Pr_T(\overline{n}, \overline{P_T(\overline{\psi(\overline{n})})}) \rightarrow \psi(\overline{n})$$

then there exists a polynomial function q such that for all sufficiently large n

$$T \vdash^{q(n)} \psi(\overline{n})$$

Finite version of Löb theorem

Conjecture (Uniform version of finite Löb theorem, UFL)

Let $T \in \mathcal{T}$ be a theory and $\psi(x)$ a formula. Then for every polynomial function p there exists a polynomial function q such that

$$\mathbb{N} \models \exists y \forall x \geq y (Pr_T(\overline{p(\dot{x})}, \overline{Pr_T(\dot{x}, \overline{P_T(\overline{\psi(\dot{x})})}) \rightarrow \psi(\dot{x})}) \rightarrow Pr_T(\overline{q(\dot{x})}, \overline{\psi(\dot{x})}))$$

Lemma

The conjecture FL implies the conjecture CON

Classical vs. polynomial provability

These conjectures indicate that there is a very close relationship between classical provability and polynomial or feasible provability. We can show these similarities in the following table.

STANDARD PROVABILITY	FEASIBLE PROVABILITY
Fast growing functions	Complexity associated with a proof
Gödel's incompleteness theorems	Finite incompleteness theorems
Löb's theorem	Finite version of Löb's theorem