On Takeuti-Yasumoto forcing

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Outline of this talk

We apply the forcing construction à la Takeuti and Yasumoto (1996) to construct generic extensions for subclasses of PTIME; $\mathsf{NC}^1$ and $\mathsf{NL}$. 
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We apply the forcing construction à la Takeuti and Yasumoto (1996) to construct generic extensions for subclasses of PTIME; $\mathsf{NC}^1$ and $\mathsf{NL}$.

Then we discuss the problem of relating separation problems in computational/proof complexity to properties of generic extensions.
Subclasses of PTIME

Many complexity classes are defined inside PTIME:

- $AC^0 =$ constant depth polynomial size Boolean circuits
- $NC^1 = ALOGTIME : O(\log n)$-time bounded alternating TMs
- $L = O(\log n)$-space bounded DTMs
- $NL = O(\log n)$-space bounded NTMs

However, no separations are known above $AC^0$:

\[ AC^0 \subsetneq NC^1 \subseteq L \subseteq NL \subseteq P \subseteq NP. \]
Two-sort Bounded Arithmetic

Two-sort language comprises
- number variables: $x, y, z, \ldots$,
- string variables: $X, Y, Z, \ldots$,
- functions and predicates: $Z(x) = 0$, $x + y$, $x \cdot y$, $|X|$, $x \in X$, $x \leq y$.

Two-sort structures consist of pairs of universes $(M_0, M)$ where $M_0$ is the number part and $M$ is the string part.

$\Sigma_0^B = \text{formulas with only bounded number quantifiers,}$

$\Sigma_i^B, \Pi_i^B$ are defined by counting the alternations of bounded string quantifier.
Two-sort theories

∀Σ^B_1-theorems of bounded arithmetic theories correspond to subclasses of PTIME:

- $V^0 = \Sigma^B_0$-COMP $\equiv AC^0$, $V^1 = \Sigma^B_1$-COMP $\equiv P$,
- $VP = V^0 +$Monotone Circuit Value $\equiv P$,
- $VNC^1 = V^0 +$Boolean Formula Value $\equiv NC^1$,
- $VNL = V^0 +$Reachability of undirected graphs $\equiv NL$,
Takeuti-Yasumoto forcing : construction of generic

Given a countable nonstandard model $\mathcal{M} = (M_0, M) \models V^1 + \neg \text{exp}$ and a Boolean algebra $B \subseteq M$ of computational objects over inputs $p_1, \ldots, p_{n-1}$ for a fixed $n \in M_0 \setminus \omega$. 
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Examples:

- $X \leq_e Y \iff \forall A \in 2^n (A \models X \rightarrow A \models Y)$.
- $X \leq_F Y \iff \mathcal{M} \models \exists P \text{ Prf}_F(P, X \rightarrow Y)$. 
Takeuti-Yasumoto forcing: construction of generic

A $M_0$-complete ideal $\mathcal{I} \subseteq \mathcal{B}$ is a nontrivial ideal such that for any $X = \langle X_0, \ldots, X_k \rangle \in M^{\mathcal{B}}$,

$$X_0, \ldots, X_k \in \mathcal{I} \Rightarrow \bigvee_{i \leq k} X_i \in \mathcal{I}$$

Elements of the ideal are ruled out from the generic extension.
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A set $\mathcal{D} \subseteq \mathbb{B}$ is dense over a $M_0$-complete ideal $\mathbb{I}$ if for any $X \in \mathbb{B} \setminus \mathbb{I}$ there is $X' \in \mathcal{D} \setminus \mathbb{I}$ such that $X' \leq X$.

A maximal filter $\mathcal{G} \subseteq \mathbb{B}$ is $\mathcal{M}$-generic over $\mathbb{I}$ if for any $\mathcal{D}$ dense over $\mathbb{I}$ and definable in $\mathcal{M}$, $\mathcal{D} \cap \mathcal{G} \neq \emptyset$. 
Takeuti-Yasumoto forcing: forcing theorem

For $X : a \to \mathcal{B}$ and $\mathcal{M}$-generic $G$, define $i_G(X) = \{ x < a : X(x) \in \mathcal{G} \}$.

$\mathcal{M}_G = \{ i_G(X) : X \in \mathcal{M}, X : a \to \mathcal{B}, \text{ for some } a \in M_0 \}$

$\mathcal{M}[G] = (M_0, M_G)$
Takeuti-Yasumoto forcing: forcing theorem

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**Theorem (Forcing Theorem)**

There is a translation $\lbrack \cdot \rbrack : \Sigma^B_0 \rightarrow \mathbb{B}$ such that for $\varphi(x, \bar{X}) \in \Sigma^B_0$, $\bar{a} \in M_0$, $A_1, \ldots, A_k \in M$, with $A_i : b_i \rightarrow \mathbb{B}$ and $\mathcal{M}$-generic $G$,

$\mathcal{M}[G] \models \varphi(\bar{a}, i_G(A_1), \ldots, i_G(A_k)) \iff \lbrack \varphi(\bar{a}, A_1, \ldots, A_k) \rbrack \in G$. 
Boolean algebras for $NC^1$

Boolean algebra for $NC^1$ is constructed by using the fact that $NC^1$ circuits are essentially Boolean formulas.

$\text{Formula}(\bar{p}) =$ the set of Boolean formulas with variables $\bar{p} = p_0, \ldots, p_{n-1}$
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Partial orders:

- $X \leq_e Y \iff \forall A \in 2^n (A \models X \to A \models Y)$.
- $X \leq_{PK} Y \iff \mathcal{M} \models \exists P \text{Prf}_{PK}(P, X \to Y)$.
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$B_{NC} = \text{Formula}(\bar{p})/\equiv_e, B_{PK} = \text{Formula}(\bar{p})/\equiv_{PK}$.
Basic Properties of generic sets

Let $\mathcal{B}$ be either $\mathcal{B}_{NC}$ or $\mathcal{B}_{PK}$.

**Lemma**

Let $I$ be an $M_0$-complete ideal. For any $X \in \mathcal{B} \setminus I$ there exists a $\mathcal{M}$-generic $G$ such that $X \in G$.

A set $S \subseteq \mathcal{B}$ is $PK$-consistent if

$$\mathcal{M} \models \text{there is no } PK\text{-proof of } \bot \text{ from } S.$$  

**Lemma**

If $S \subseteq \mathcal{B}$ is $PK$-consistent then there exists a $\mathcal{M}$-generic $G$ such that $S \subseteq G$. 

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Boolean algebra for $NL$

Boolean algebra for $NL$ is obtained by extending propositional formulas to allow transitive closure operators.

**TC connective:** $TC_{n,a,b}^k(p_0,1,\ldots,p_{n-1,n-2})$.

**Intended meaning:** there is a path of length $\leq k$ from $a$ to $b$ in the graph defined by $p_0,1,\ldots,p_{n-1,n-2}$.

**Partial orders:**

- $X \leq_e Y \iff \forall A \in 2^n (A \models X \rightarrow A \models Y)$.
- $X \leq_{PTCK} Y \iff M \models \exists P \text{ Prf}_{PTCK}(P,X \rightarrow Y)$.

**NB:** $PTCK$ is $PK$ extended by axioms for TC connectives.

$\mathbb{B}_{TC} = \text{Formula}(\bar{p})/ =_e$, $\mathbb{B}_{PTCK} = TCF(\bar{p})/ =_{PTCK}$
Generic models for $NC^1$ and $NL$

**Theorem**

1. If $G \subseteq B_{NC}$ or $B_{PK}$ is $M$-generic then $M[G] \models VNC^1$.
2. If $G \subseteq B_{TC}$ or $B_{PTCK}$ is $M$-generic then $M[G] \models VNL$. 

Proof is given by constructing Boolean formulas/TC formulas witnessing Boolean formula value problem/st-connectivity resp.
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Proof is given by constructing Boolean formulas/TC formulas witnessing Boolean formula value problem/st-connectivity resp.
Theorem (K)

Let $\mathcal{M} \models V^1$. $\mathcal{M} \models P \subseteq NC^1$ if and only if $\mathcal{M}[G] \models VP$ for any $\mathcal{M}$-generic $G \subseteq B_{NC}$. 

(From left to right).

If $\mathcal{M} \models P \subseteq NC^1$ then we can witness Monotone Circuit Value by Boolean formulas.

(From right to left).

If $\mathcal{M} \models P \not\subseteq NC^1$ then we can construct a PK-consistent set violating MCV.
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Theorem (K)

Let $\mathcal{M} \models V^1$ and suppose that the following conditions hold in $\mathcal{M}$.

- $G_1^* \leq_p G_0$.
- Propositional translations of $\Sigma^B_1$-theorems of $V^1$ have polynomial size $G_1^*$ proofs.

Then $\mathcal{M}[G] \models VP$ for any $\mathcal{M}$-generic $G \subseteq \mathcal{B}_F$.

Theorem (K)

Let $\mathcal{M} \models V^1$ and suppose that $\mathcal{M} \models EF \ntriangleleft_p F$. Then there exists an $\mathcal{M}$-generic $G \subseteq \mathcal{B}_F$ such that

$$\mathcal{M}[G] \not\models V^1.$$
Problems

- Construct Boolean algebras for other complexity classes; eg. three-sort models for PSPACE.

- Develop techniques to decide whether a given $\Sigma_2^B \cup \Pi_2^B$ formula holds or not in generic extensions.