On the proof complexity in two universal proof system for all versions of many-valued logics

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Formerly two types of propositional proof systems were described in

Anahit Chubaryan, Artur Khamisyan, Two types of universal proof systems for all variants of many-valued logics and some properties of them, Iran Journal of Computer Science, 2018, Springer Verlag

such that propositional proof system for every version of MVL can be presented in both of described forms. We call these systems an Universal Systems.

The first of introduced systems (US) is a Gentzen-like system, the second one (UE) is based on the generalization of the notion of determinative disjunctive normal form. The last type proof systems are “weak” ones with a “simple strategist” of proof search and we have investigated the quantitative properties, related to proof complexity characteristics in them. Now we investigate the relations between the main proof complexty measures in both universal systems.

Let us recall some notions and notations

Main notions of k-valued logic.

Let E_k be the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$. We use the well-known notions of propositional formula, which defined as usual from propositional variables with values from E_k, (may be also propositional constants), parentheses (,), and logical connectives $\&, \vee, \supset, \neg$, every of which can be defined by different mode. Additionaly we use two modes of exponential function p^{σ} and introduce the additional notion of formula: for every formulas A and B the expression $\boldsymbol{A^B}$ (for both modes) is formula also.

In the considered logics either only **1** or every of values $\frac{1}{2} \leq \boldsymbol{\frac{i}{k-1}} \leq$ 1 can be fixed as ***designated values***.

Definitions of main logical functions are:

(1) $\boldsymbol{p} \vee \boldsymbol{q} = max(p, q)$ or (2) $\boldsymbol{p} \vee \boldsymbol{q} = \min(\mathrm{p} + \mathrm{q}, 1)$,

(1) $\boldsymbol{p} \& q = min(p, q)$ or (2) $\boldsymbol{p} \& q = \max(\boldsymbol{p} + q - 1, 0)$

For implication we have two following versions:

(1) $\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, & for\ \ p \leq q \\ 1 - p + q, & for\ \ p > q \end{cases}$ or

(2) $\mathbf{p} \supset \boldsymbol{q} = \begin{cases} 1, & for & p \leq q \\ q, & for & p > q \end{cases}$

And for negation two versions also:

$(\mathbf{1})$ $\neg \boldsymbol{p} = 1 - p$ or (2) $\neg \boldsymbol{p} = ((k-1)p + 1)(mod\ k)/(k-1)$

For propositional variable $\boldsymbol{p}$ and $\boldsymbol{\delta} = \frac{i}{k-1} (0 \le i \le k\text{-}1)$ we define additionally "exponent" functions:

(1) $\boldsymbol{p}^{\boldsymbol{\delta}} = (p \supset \delta) \& (\delta \supset p)$ with (1) implication and

(2) $\boldsymbol{p}^{\boldsymbol{\delta}}$ as p with $(k\text{-}1)(1 - \delta)$ (2) negations.

If we fix "**1**" (every of values $\frac{1}{2} \leq \frac{i}{\text{k}-1} \leq 1$) as designated value, so a formula φ with variables $p_1, p_2, \ldots p_n$ is called ***1-k*-tautology** (***≥1/2-k-tautology***) if for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ assigning δ_j ($1 \leq j \leq n$) to each p_j gives the value 1 (or some value $\frac{i}{\text{k}-1} \geq \frac{1}{2}$) of φ.

Sometimes we call ***1-k*-tautology** or ***≥1/2-k-tautology*** simply ***k*-tautology.**

Determinative Disjunctive Normal Form for MVL

The notions of determinative conjunct is defined for all variants of MVL in above paper.

For every propositional variable p in k-valued logic

$$p^0, p^{1/\mathrm{k}-1}, \ldots, p^{\mathrm{k}-2/\mathrm{k}-1} \text{ and } p^1$$

in sense of both exponent modes are the ***literals.*** The conjunct K (term) can be represented simply as a set of literals (no conjunct contains a variable with different measures of exponents simultaneously).

Let φ be a propositional formula of k-valued logic, $P = \{p_1, p_2, \dots, p_n\}$ be the set of all variables of φ and $P^{'} = \{p_{i_1}, p_{i_2}, \dots, p_{i_m}\}$ $(1 \leq m \leq n)$ be some subset of P.

Short **Definition 1.** Given $\tilde{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_m) \in E_{\mathrm{k}}^m$, the conjunct $K^\sigma = \{{p_{i_1}}^{\sigma_1}, {p_{i_2}}^{\sigma_2}, \dots, {p_{i_m}}^{\sigma_m}\}$ is called $\varphi - \frac{i}{k-1}$-determinative $(0 \leq i \leq k-1)$, if assigning σ_j $(1 \leq j \leq m)$ to each p_{i_j} we obtain the value $\frac{i}{k-1}$ of φ independently of the values of the remaining variables.

Every $\varphi - \frac{i}{k-1}$ −determinative conjunct is called also φ-determinative or determinative for φ.

Definition 2.. A disjunctive normal form (DNF) $D = \{K_1, K_2, \ldots, K_j\}$ is called determinative DNF (DDNF) for φ if $\varphi = D$ and if "1" (every of values $\frac{1}{2} \leq \frac{i}{\mathrm{k}-1} \leq 1$) is (are) fixed as designated value, then every conjunct $K_i (1 \leq i \leq j)$ is 1-determinative ($\frac{i}{\mathrm{k}-1}$ − determinative from indicated intervale) for φ.

Definitions of universal systems for MVL .

The universal elimination system UE for all versions of MVL.

The axioms of Elimination systems **UE** aren't fixed, but for every formula $\boldsymbol{k-valued\ \varphi}$ each conjunct from some DDNF of $\boldsymbol{\varphi}$ can be considered as an axiom.

For *k*-valued logic the inference rule is *elimination rule* (ε-rule)

$$\frac{K_0 \cup \{p^0\},\ \ K_1 \cup \left\{p^{\frac{1}{k-1}}\right\},\ \ \dots,\ \ K_{k-2} \cup \left\{p^{\frac{k-2}{k-1}}\right\},\ \ K_{k-1} \cup \{p^1\}}{K_0 \cup K_1 \cup \dots \cup K_{k-2} \cup K_{k-1}},$$

where mutual supplementary literals (variables with corresponding (1) or (2) exponents) are eliminated.

A finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of **UE** or is inferred from earlier conjuncts in the sequence by ε-rule is called a proof in **UE**. A DNF $D = \{K_1, K_2, \ldots, K_l\}$ is k-tautological if by using ε-rule can be proved the empty conjunct ($\emptyset$) from the axioms $\{K_1, K_2, \ldots, K_l\}$.

The completeness of these systems is obvious.

Sequent type system US for all versions of MVL.

Sequent system uses the denotation of sequent $\Gamma \vdash \Delta$ where Γ (antecedent) and Δ (succedent) are finite (may be empty) sequences (or sets) of propositional formulas.

For every literal C and for any set of literals K the axiom sxeme of propositional system **US** is $K, C \vdash C$.

For every formulas A , B, for any sets of literals K, K_i (i = 0, ... , k − 1), each σ_1, σ_2, σ from the set E_k and for $* \in \{\&, \vee, \supset\}$ the logical rules of **US** are:

$$\vdash * \frac{\mathrm{K} \vdash A^{\sigma_1} \text{ and } \qquad \mathrm{K} \vdash B^{\sigma_2}}{\mathrm{K} \vdash (A * B)^{\varphi_*(A,B,\sigma_1,\sigma_2)})}$$

$$\vdash \exp \frac{\mathrm{K} \vdash A^{\sigma_1} \text{ and } \qquad \mathrm{K} \vdash B^{\sigma_2}}{\mathrm{K} \vdash \left(A^B\right)^{\varphi \exp(A,B,\sigma_1,\sigma_2)})}$$

$$\vdash \neg \frac{\mathrm{K} \vdash A^{\sigma}}{\mathrm{K} \vdash \quad (\neg A)^{\varphi_\neg(A,\sigma)}}$$

$$\text{literals elimination} \vdash \frac{K_0, p^0 \vdash A, \quad K_1, p^{\frac{1}{k-1}} \vdash A, \quad \ldots, \quad K_{k-2}, p^{\frac{k-2}{k-1}} \vdash A, \quad K_{k-1}, p^1 \vdash A}{K_0 \cup K_1 \cup \ldots \cup K_{k-2} \cup K_{k-1} \vdash A},$$

where many-valued functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{\exp}(A, B, \sigma_1, \sigma_2)$, $\varphi_\neg(A, \sigma)$, must be defined individually for each version of MVL such, that

1) formulas $A^{\sigma_1} \supset (B^{\sigma_2} \supset (A * B)^{\varphi_*(A,B,\sigma_1,\sigma_2)})$, $A^{\sigma_1} \supset (B^{\sigma_2} \supset (A^B)^{\varphi_{\exp}(A,B,\sigma_1,\sigma_2)})$ and

$A^{\sigma} \supset (\neg A)^{\varphi_\neg(A,\sigma)}$ must be *k*-tautolog*y* in this version,

2) if for some $\sigma_1, \sigma_2, \sigma$ the value of $\sigma_1 * \sigma_2$ (${\sigma_1}^{\sigma_2}$, $\neg\sigma$) is one of ***designed values*** in this version of MVL , then $(\sigma_1 * \sigma_2)^{\varphi_*(\sigma_1,\sigma_2,\sigma_1,\sigma_2)} = \sigma_1 * \sigma_2$ $(({\sigma_1}^{\sigma_2})^{\varphi_{\exp}(\sigma_1,\sigma_2,\sigma_1,\sigma_2)} = {\sigma_1}^{\sigma_2}, (\neg\sigma)^{\varphi_\neg(\sigma,\sigma)} = \neg\sigma)$.

We use the well known notion of proof in sequent systems. We say that formula ***A is derived in US iff the sequent*** $\vdash$ ***A is deduced in US.***

Completeness of US is proved in above paper.

Proof complexity measures

In the theory of proof complexity four main characteristics of the proof are: *t-complexity (time),* defined as the number of proof steps, *l-complexity (size)*, defined as total number of proof symbols , *s-complexity* (space), informal defined as maximum of minimal number of symbols on blackboard needed to verify all steps in the proof and *w-complexity (width*), defined as the maximum of widths of proof formulas.

Let Φ be a proof system and φ be a tautology. We denote by $t_\varphi^\Phi(l_\varphi^\Phi, s_\varphi^\Phi, w_\varphi^\Phi)$ the minimal possible value of *t-complexity* (*l-complexity*, *s-complexity*, *w-complexity) f*or all proofs of tautology φ in Φ.

For some class of many-valued tautologies simultaneously optimal bounds (asymptotically the same upper and lower bounds) for each of main proof complexity characteristics were obtained for the system UE of some versions of many-valued logic in

A.A.Chubaryan, A.S.Tshitoyan, A.A.Khamisyan,On some proof systems for many-valued logics and on proof complexities in it, (in Russian) National Academy of Sciences of Armenia, Reports, Vol.116, N2, 2016, 108-114.

In order to obtain the same bounds in US we investigate the relations between the main proof complexty measures in both universal systems.

Let Φ_1 and Φ_2 be two different proof systems.

Definition 3. The system Φ_1 *p-simulates* the system Φ_2 if there exist the polynomial *p()* such, that for each formula φ provable both in the systems Φ_1 and Φ_2, we have $l_{\varphi}^{\Phi 1} \leq p(l_{\varphi}^{\Phi 2})$.

Definition 4. The systems Φ_1 and Φ_2 are *p-equivalent*, if systems Φ_1 and Φ_2 *p-simulate* each other.

We show that for every *k-tautology A*

$$\text{a)}\ \boldsymbol{t_A^{UE}} \le \boldsymbol{t_A^{US}}, \quad \boldsymbol{l_A^{UE}} \le \boldsymbol{l_A^{US}},$$

$$\boldsymbol{s_A^{UE}} \le \boldsymbol{s_A^{US}}, \quad \boldsymbol{w_A^{UE}} \le \boldsymbol{w_A^{US}},$$

so, the system UE *p-simulates* the system US,

$$\text{b)}\ \boldsymbol{t_A^{US}} \le \boldsymbol{t_A^{UE}|A|}, \ \boldsymbol{l_A^{US}} \le \boldsymbol{l_A^{UE}|A||A|},$$

$$\boldsymbol{s_A^{US}} \le \boldsymbol{s_A^{UE}|A|}, \quad \boldsymbol{w_A^{US}} \le \boldsymbol{w_A^{UE}|A|},$$

where by $\boldsymbol{|A|}$ the size of A is denoted.

Note that there are many sequences of k-tautologies A_n , sizes of which can be very long, but $t - complexities$ of their UE-proofs are bounded by some constant, therefore the system US does not *p-simulate* the system UE and the systems UE and US do not be *p-equivalent* , but nevertheless some classes of k-tautologies have the same proof complexities bounds in both systems.

Bounds of proof complexity measures of some classes of $\boldsymbol{k}$-tautologies in some variants of $\boldsymbol{UE}$ and US.

In some papers in area of propositional proof complexity for 2-valued classical logic the following tautologies (Topsy-Turvy Matrix) play key role

$$\boldsymbol{TTM_{n,m}} = \bigvee_{(\sigma_1,\sigma_2,\dots,\sigma_n)\in E^n} \bigwedge_{j=1}^{m} \bigvee_{i=1}^{n} p_{i_j}{}^{\sigma_j} \; (\boldsymbol{n \geq 1, 1 \leq m \leq 2^n - 1}).$$

For all fixed $\boldsymbol{n \geq 1}$ and $\boldsymbol{m}$ in above indicated intervals every formula of this kind expresses the following true statement: given a 0,1-matrix of order $\boldsymbol{n \times m}$ we can “topsy-turvy” some strings (writing 0 instead of 1 and 1 instead of 0) so that each column will contain at least one 1.

In Arman Tshitoyan, 2017, Bounds of proof complexities in some systems for many-valued logics, Isaac Scientific Publishing (ISP), Journal of Advances in Applied Mathematics, Vol. 2, No. 3, July, 164-172.

the notion "topsy-turvy" is generalized as follow:

Definition 5. Given $\tilde{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_m) \in E_k^m$ and **δ** $=\frac{i}{\mathrm{k}-1}$ ($0 \le i \le k$-1) we call **δ**-(1)-topsy-turvy-result (**δ**-(2)-topsy-turvy-result) the cortege $\widetilde{\sigma\delta}$, which contains every σ_j ($1 \le j \le m$) **with (1) exponent δ for (1) negation** (**with (2) exponent δ for *(2) negation***) .

On the base of this notion many k-tautologies were described and their proof complexities measures was investigated in UE systems for some variants of MVL

Main Theorem.

1) In US and UE systems for MVL with (1) conjunction, (1) or (2) disjunction, (1) implication, (1) negation ((1) conjunction, (1) disjunction, (2) implication, (2) negation) for 1-k-tautologies (k≥3) $\boldsymbol{\varphi_n} = \mathbf{TTM_{n,m}}$ for every $\mathbf{n \geq 1}$ and $\mathbf{m = k^{[n/k]}}$, where $\mathbf{TTM_{n,m}} = \bigvee_{(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \ldots, \boldsymbol{\sigma_n}) \in \mathbf{E_k^n}} \bigwedge_{\mathbf{j=1}}^{\mathbf{m}} \bigvee_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{p_{i_j}}^{\boldsymbol{\sigma_j}}$ the following bounds are true

$$log_k log_k\big(t(\varphi_n)\big) = \theta(n);$$

$$log_k log_k\big(l(\varphi_n)\big) = \theta(n);$$

$$log_k\big(s(\varphi_n)\big) = \theta(n);$$

$$log_k\big(w(\varphi_n)\big) = \theta(n).$$

2) In US and UE systems for MVL with (1) conjunction, (1) or (2) disjunction, (1) implication, (1) negation for ≥1/2-k-tautologies (k≥3) $\boldsymbol{\varphi_n}$=$\mathbf{LTTM_{n,m}}$, for every $\mathbf{n \geq 1}$ and and $\mathbf{m \leq 2^n - 1}$ where $\mathbf{LTTM_{n,m}} = \bigvee_{(\boldsymbol{\sigma_1},\boldsymbol{\sigma_2},\ldots,\boldsymbol{\sigma_n})\in \mathbf{E^n}} \bigwedge_{\mathbf{j=1}}^{\mathbf{m}} \bigvee_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{p_{i_j}}^{\boldsymbol{\sigma_j}}$, where E={0,1} , the following bounds are true

$$log_2 log_k\big(t(\varphi_n)\big) = \theta(n);$$

$$log_2 log_k\big(l(\varphi_n)\big) = \theta(n);$$

$$log_2\big(s(\varphi_n)\big) = \theta(n);$$

$$log_2\big(w(\varphi_n)\big) = \theta(n).$$

3) In US and UE systems for MVL with (1) conjunction, (1) or (2) disjunction, (2) implication, (2) negation for ≥1/2-k-tautologies (k≥3) $\boldsymbol{\varphi_n}$=$\mathbf{GTTM_{n,m}}$, for every $\mathbf{n \geq 1}$ and and $\mathbf{m} \leq \mathrm{k}^{\mathbf{n}} - \mathbf{1}$ where $\mathrm{G}\mathbf{TTM_{n,m}} = \bigvee_{(\sigma_1,\sigma_2,\ldots,\sigma_n)\in E_k^n} \bigwedge_{j=1}^{m} \bigvee_{i=1}^{n} \mathbf{p_{i_j}}^{\sigma_j}$ the following bounds are true

$$log_k log_k\big(t(\varphi_n)\big) = \theta(n);$$

$$log_k log_k\big(l(\varphi_n)\big) = \theta(n);$$

$$log_k\big(s(\varphi_n)\big) = \theta(n);$$

$$log_k\big(w(\varphi_n)\big) = \theta(n).$$

The analogous bounds of proof complexity measures can be obtained in US and UE type systems for all variants of MVL. The preference of such systems is the simple strategy of proof steps choice and the possibility of the automatic receipt of exponential lower bounds for tautologies with specific properties: minimal numbers of literals in determinative conjunct must be by order nearly equal to the size of formula.

Thank you for attention