An existence result for a class of partial differential equations

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The notion of solution for partial differential equations is not unique.

Transport equation in dimension 1

\[
\begin{align*}
  u_t(t, x) &= u_x(t, x), \quad x \in \Omega \subseteq \mathbb{R}, \quad t \geq 0; \\
  u(0, x) &= u_0(x), \quad x \in \Omega.
\end{align*}
\]

Solution in the sense of distributions: \( u(t, x) = u_0(x - t) \).
Nonlinear diffusion - cubic-like (Plotnikov 1994)

\[
\begin{align*}
    u_t(t, x) &= \Delta \phi(u(t, x)), \quad x \in \Omega, \quad t \geq 0; \\
    u(0, x) &= u_0(x), \quad x \in \Omega
\end{align*}
\]

with \( \phi \) of the form

Solution in the sense of Young measures.
Partial differential equations - III

Nonlinear diffusion - Perona–Malik (Smarrazzo 2008)

\[
\begin{cases}
  u_t(t, x) = \Delta \phi(u(t, x)), & x \in \Omega \subseteq \mathbb{R}^k, \; t \geq 0; \\
  u(0, x) = u_0(x), & x \in \Omega
\end{cases}
\]

with \( \phi \) of the form

Solution: the sum of a Young measure and of a Radon measure.
Nonstandard analysis in a nutshell

Let $\mathcal{P}(X)$ be the power set of $X$, and define the superstructure over $X$ as

$$\mathbb{V}(X) = \bigcup_{n \in \mathbb{N}} \mathcal{P}^n(X).$$

**Definition (Nonstandard universe)**

A nonstandard universe is a triple $(\mathbb{V}(\mathbb{R}), \mathbb{V}(\ast\mathbb{R}), \ast)$ such that:

- $\mathbb{V}(\mathbb{R})$ and $\mathbb{V}(\ast\mathbb{R})$ are superstructures;
- $\ast : \mathbb{V}(\mathbb{R}) \rightarrow \mathbb{V}(\ast\mathbb{R})$ maps $\mathbb{R}$ properly into $\ast\mathbb{R}$ (i.e. $\mathbb{R} \neq \ast\mathbb{R}$);
- for each first order formula $\phi(X_1, \ldots, X_k)$ with bounded quantifiers and with $X_1, \ldots, X_k \in \mathbb{V}(\mathbb{R})$,

$$\phi(X_1, \ldots, X_n) \iff \phi(\ast X_1, \ldots, \ast X_n).$$
Grid functions

Definition

Let \( N_0 \in \star \mathbb{N} \) be an infinite hypernatural number. Set \( N = N_0! \), \( \varepsilon = 1/N \) and define \( X = \{ n\varepsilon : -N^2 \leq n \leq N^2 \} \).

Proposition

Let \( \Omega \subseteq \mathbb{R}^k \) be an open set, and define \( \Omega_X = \star \Omega \cap X^k \). Then \( \circ \Omega_X = \overline{\Omega} \).

Definition (Grid functions)

The algebra of grid functions defined over \( \Omega_X \) is

\[
\mathcal{G}(\Omega_X) = \{ f : \Omega_X \to \star \mathbb{R} \text{ and } f \text{ is internal} \}.
\]
Grid derivative

Definition (Grid derivative)

For a function $f \in \mathbb{G}(\Omega_x)$, we define the grid derivative $D_i f$ as

$$D_i f(x) = \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}.$$ 

If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index,

$$D^\alpha f(x) = D_1^{\alpha_1} \ldots D_k^{\alpha_k} f(x).$$
Grid functions and distributions

Theorem (B. 2017)

For every distribution $T \in \mathcal{D}'(\Omega)$ there exists a grid function $f$ such that

$$
\circ \left( \sum_{x \in \Omega} f(x) \ast \varphi(x) \right) = \langle T, \varphi \rangle_{\mathcal{D}(\Omega)}.
$$

In this case, we will say that $[f] = T$.
Moreover, the grid derivative represents the distributional derivative, i.e.

if $[f] = T$, then $[\mathbb{D}f] = T'$. 
Grid functions and Young measures

Theorem (B. 2017)

For every grid function $f$ there exists a parametrized measure $\nu$ over $\Omega$ such that, if we define $g(\nu) : \Omega \to \mathbb{R}$ by

$$g(\nu)(x) = \left(\int_{\mathbb{R}} gd\nu_x\right)$$

for every $x \in \Omega$,

then $[*g(f)] = g(\nu)$.

In the above formulas, $g \in C^0(\mathbb{R})$ and $\lim_{|x| \to \infty} g(x) = 0$.

Proposition (Cutland 1986 & B. 2017)

For any $f \in \mathbb{G}(\Omega_X)$ such that $\|f\|_\infty$ is finite, $\nu$ is a Young measure and $[f]$ is its barycentre.
The existence result

Preliminary definitions

Differential operators

Let
\[ P(x_1, \ldots, x_k) = \sum_{|\alpha| \leq d} a_\alpha x_1^{\alpha_1} \ldots x_k^{\alpha_k}, \]
be a polynomial in \( k \) variables with \( a_\alpha \in C^\infty([0, +\infty) \times \Omega) \).

Define
\[ P_{\partial} = P \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right) = \sum_{|\alpha| \leq d} a_\alpha \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \ldots \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}} \]
and
\[ P_D = P \left( D_1, \ldots, D_k \right) = \sum_{|\alpha| \leq d} *a_\alpha D_1^{\alpha_1} \ldots D_k^{\alpha_k}. \]
Let $P, P_1, \ldots, P_j$ be polynomials in $k$ variables with coefficients in $C^\infty([0, +\infty) \times \Omega)$. Let also

\begin{align*}
\left\{ \begin{array}{ll}
    u_t = P \partial f (P_1, \partial u, \ldots, P_j, \partial u) & \text{in } \Omega \subseteq \mathbb{R}^k, \\
    u(0, x) = u_0(x),
\end{array} \right.
\end{align*}

and

\begin{align*}
\left\{ \begin{array}{ll}
    u_t = P \mathbb{D}^* f (P_1, \mathbb{D} u, \ldots, P_j, \mathbb{D} u) & \text{in } \Omega_x, \\
    u(0, x) = ^* u_0(x).
\end{array} \right.
\end{align*}

If $f$ is locally Lipschitz, then problem (2) has a unique solution $u \in ^* C^1([0, T], \mathbb{G}(\Omega_x))$. 
The existence result - II

Existence of a real solution (B. 2018)

Let \( u \) be a solution to problem (2) and let \( \nu^i \) be the parametrized measure corresponding to \( P_i, D u, \ i = 1, \ldots, j \).

Then \([u]\) and \(\nu^1, \ldots, \nu^j\) satisfy

\[
\int_0^T \langle [u], \varphi_t \rangle_{\mathcal{D}(\Omega)} + \langle f(\nu^1, \ldots, \nu^j), P_i^\dagger \varphi \rangle_{\mathcal{D}(\Omega)} dt +
+ \int_{\Omega} u_0(x) \varphi(0, x) dx = 0.
\]

for all \( \varphi \in C^1([0, T], C^\infty(\Omega)) \) with \( \varphi(T, x) = 0 \) for all \( x \in \Omega \).
The grid function formulation for the transport equation

\begin{equation*}
\begin{aligned}
\begin{cases}
  u_t = D(u) \text{ in } \Omega_x \\
  u(0, x) = u_0(x).
\end{cases}
\end{aligned}
\end{equation*}

The solution to this problem is \( u(t) = e^{-tD} u_0(x) \) for every (distributional, measure-valued, ...) initial data \( u_0 \).

Meaning of the solution

This is a generalization of the equality \( e^{-tD} f(x) = f(x - t) \) for analytic functions.
The grid function formulation for the nonlinear diffusion

The grid function formulation

We discretize the Laplacian by using the grid derivative:

$$\Delta_{X}^{\ast} \phi(u(x, t)) = \sum_{i=1}^{k} D_i D_i (\ast \phi(u(x - \varepsilon e_i, t))),$$

and impose Neumann boundary conditions by a first order finite-difference approximation.

We obtain the hyperfinite system of ODEs

$$\begin{align*}
&\begin{cases}
  u_t = \Delta_{X}^{\ast} \phi(u), & x \in \Omega_{X}, \ t \geq 0; \\
  u(0, x) = \ast u_0(x), & x \in \Omega_{X},
\end{cases}
\end{align*}$$
Properties of the grid solution

Existence and uniqueness of the grid solution (B. 2017)

The grid function formulation has a unique global solution $u : \mathbb{R}_0^+ \rightarrow \Omega_X$. Moreover, the mass of the solution is preserved:

$$\|u(t, x)\|_1 = \|u_0\|_1 \text{ for all } t \geq 0.$$ 

Coherence with the measure-valued solutions (B. 2017)

$[u]$ is a non-negative Radon measure, and $[\phi(u)] \in L^\infty(\Omega \times [0, T])$ for all $T \geq 0$. Moreover, it satisfies an entropy condition that characterizes admissible solutions to the original problem.
Conclusions

The strength of nonstandard analysis

None of this would be possible if the function $^*$ did not preserve first order properties!

Perspectives for future research

- Study classically ill-posed PDEs with grid functions (relevant also for many mathematical models of physical phenomena);
- develop a nonlinear theory of distributions;
- study the relations between grid functions, Colombeau’s algebras, algebras of asymptotic functions, and spaces of ultrafunctions.


Young measures

Definition (Young 1937, . . .)

A Young measure over $\Omega$ is a measurable function $\nu : \Omega \rightarrow P(\mathbb{R})$. The “composition” between a function $g \in C^0_b(\mathbb{R})$ and $\nu$ is the function defined a.e. by

$$g(\nu)(x) = \int_{\mathbb{R}} gd\nu_x.$$  

Theorem (Prokhorov 1956, Ball 1989, . . .)

Let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions, uniformly bounded in $L^p(\Omega)$ for some $1 \leq p \leq \infty$. Then there exists a Young measure $\nu$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} g(z_n(x))\varphi(x)dx = \int_{\Omega} \left( \int_{\mathbb{R}} g d\nu_x \right) \varphi(x)dx$$

for all $g \in C^0_b(\mathbb{R})$ and for all $\varphi \in L^1(\Omega)$. 

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Grid functions and Young measures

**Theorem**

For every $f \in \mathcal{G}(\Omega_X)$, there exists a measurable $\nu : \Omega \to \mathcal{M}(\mathbb{R})$ such that

$$
\circ \left( \sum_{x \in \Omega_X^*} *g(f(x)) *\varphi(x) \right) = \int_{\Omega} \left( \int_{\mathbb{R}} gd\nu_x \right) \varphi(x) dx
$$

for all $g \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0(\mathbb{R})$. Moreover, for all $x \in \Omega$ and for all Borel $A \subseteq \mathbb{R}$, $0 \leq \nu_x(A) \leq 1$.

**Proposition (Cutland 1986 & B. 2017)**

For every $f \in \mathcal{G}(\Omega_X)$ such that $\|f\|_\infty$ is finite, $\nu$ is a Young measure and $[f]$ is its barycentre.