An existence result for a class of partial differential equations

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Partial differential equations - I

Problem

The notion of *solution* for partial differential equations is not unique.

Transport equation in dimension 1

$$\begin{cases} u_t(t,x) = u_x(t,x), & x \in \Omega \subseteq \mathbb{R},\ t \geq 0; \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

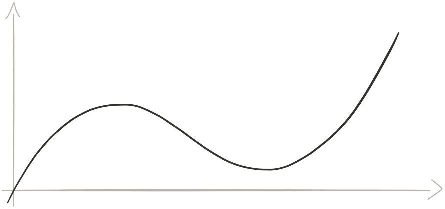
Solution in the sense of distributions: $u(t,x) = u_0(x-t)$.

Partial differential equations - II

Nonlinear diffusion - cubic-like (Plotnikov 1994)

$$\begin{cases} u_t(t,x) = \Delta\phi(u(t,x)), & x \in \Omega,\ t \geq 0; \\ u(0,x) = u_0(x), & x \in \Omega \end{cases}$$

with ϕ of the form



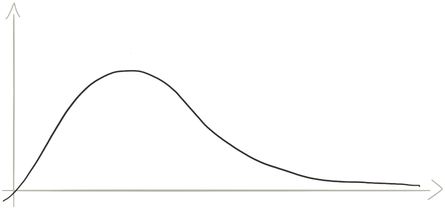
Solution in the sense of Young measures.

Partial differential equations - III

Nonlinear diffusion - Perona–Malik (Smarrazzo 2008)

$$\begin{cases} u_t(t,x) = \Delta\phi(u(t,x)), & x \in \Omega \subseteq \mathbb{R}^k,\ t \geq 0; \\ u\,(0,x) = u_0(x), & x \in \Omega \end{cases}$$

with ϕ of the form



Solution: the sum of a Young measure and of a Radon measure.

Nonstandard analysis in a nutshell

Let $\mathcal{P}(X)$ be the power set of X, and define the superstructure over X as

$$\mathbb{V}(X) = \bigcup_{n\in\mathbb{N}} \mathcal{P}^n(X).$$

Definition (Nonstandard universe)

A nonstandard universe is a triple $(\mathbb{V}(\mathbb{R}), \mathbb{V}({}^*\mathbb{R}), {}^*)$ such that:

- $\mathbb{V}(\mathbb{R})$ and $\mathbb{V}({}^*\mathbb{R})$ are superstructures;
- ${}^* : \mathbb{V}(\mathbb{R}) \to \mathbb{V}({}^*\mathbb{R})$ maps $\mathbb{R}$ properly into ${}^*\mathbb{R}$ (i.e. $\mathbb{R} \neq {}^*\mathbb{R}$);
- for each first order formula $\phi(X_1, \dots, X_k)$ with bounded quantifiers and with $X_1, \dots, X_k \in \mathbb{V}(\mathbb{R})$,

$$\phi(X_1, \dots, X_n) \Longleftrightarrow \phi({}^*X_1, \dots, {}^*X_n).$$

Grid functions

Definition

Let $N_0 \in {}^*\mathbb{N}$ be an infinite hypernatural number. Set $N = N_0!$, $\varepsilon = 1/N$ and define $\mathbb{X} = \{n\varepsilon : -N^2 \leq n \leq N^2\}$.

Proposition

Let $\Omega \subseteq \mathbb{R}^k$ be an open set, and define $\Omega_{\mathbb{X}} = {}^*\Omega \cap \mathbb{X}^k$. Then ${}^\circ\Omega_{\mathbb{X}} = \overline{\Omega}$.

Definition (Grid functions)

The algebra of grid functions defined over $\Omega_{\mathbb{X}}$ is

$$\mathbb{G}(\Omega_{\mathbb{X}}) = \{f : \Omega_{\mathbb{X}} \to {}^*\mathbb{R} \text{ and } f \text{ is internal}\}.$$

Grid derivative

Definition (Grid derivative)

For a function $f \in \mathbb{G}(\Omega_{\mathbb{X}})$, we define the grid derivative $\mathbb{D}_i f$ as

$$\mathbb{D}_i f(x) = \frac{f(x+\varepsilon e_i) - f(x)}{\varepsilon}.$$

If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index,

$$\mathbb{D}^\alpha f(x) = \mathbb{D}_1^{\alpha_1} \ldots \mathbb{D}_k^{\alpha_k} f(x)$$

Grid functions and distributions

Theorem (B. 2017)

For every distribution $T \in \mathscr{D}'(\Omega)$ there exists a grid function f such that

$$^{\circ}\left(\sum_{x \in \Omega_{\mathbb{X}}} f(x)\, {}^{*}\varphi(x) \right) = \langle T, \varphi \rangle_{\mathscr{D}(\Omega)}.$$

In this case, we will say that $[f] = T$.
Moreover, the grid derivative represents the distributional derivative, i.e.

$$\text{if } [f] = T, \text{ then } [\mathbb{D}f] = T'.$$

Grid functions and Young measures

Theorem (B. 2017)

For every grid function f there exists a parametrized measure ν over Ω such that, if we define $g(\nu) : \Omega \to \mathbb{R}$ by

$$g(\nu)(x) = \left(\int_{\mathbb{R}} g d\nu_x \right) \text{ for every } x \in \Omega,$$

then $[{}^*g(f)] = g(\nu)$.
In the above formulas, $g \in C^0(\mathbb{R})$ and $\lim_{|x|\to\infty} g(x) = 0$.

Proposition (Cutland 1986 & B. 2017)

For any $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ such that $\|f\|_\infty$ is finite, ν is a Young measure and $[f]$ is its barycentre

Preliminary definitions

Differential operators

Let

$$P(x_1, \dots, x_k) = \sum_{|\alpha| \leq d} a_\alpha x_1^{\alpha_1} \dots x_k^{\alpha_k},$$

be a polynomial in k variables with $a_\alpha \in C^\infty([0, +\infty) \times \Omega)$.
Define

$$P_\partial = P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) = \sum_{|\alpha| \leq d} a_\alpha \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}$$

and

$$P_\mathbb{D} = P\left(\mathbb{D}_1, \dots, \mathbb{D}_k\right) = \sum_{|\alpha| \leq d} {}^*a_\alpha \mathbb{D}_1^{\alpha_1} \dots \mathbb{D}_k^{\alpha_k}.$$

The existence result - I

The nonstandard existence result (B. 2018)

Let $P, P_1, \dots, P_j$ be polynomials in k variables with coefficients in $C^\infty([0,+\infty) \times \Omega)$. Let also

$$\begin{cases} u_t = P_\partial f\left(P_{1,\partial} u, \dots, P_{j,\partial} u\right) \quad \text{in } \Omega \subseteq \mathbb{R}^k, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

and

$$\begin{cases} u_t = P_\mathbb{D} {}^*f\left(P_{1,\mathbb{D}} u, \dots, P_{j,\mathbb{D}} u\right) \quad \text{in } \Omega_\mathbb{X}, \\ u(0,x) = {}^*u_0(x). \end{cases} \tag{2}$$

If f is locally Lipschitz, then problem (2) has a unique solution $u \in {}^*C^1([0,T], \mathbb{G}(\Omega_\mathbb{X}))$.

The existence result - II

Existence of a real solution (B. 2018)

Let u be a solution to problem (2) and let ν^i be the parametrized measure corresponding to $P_{i,\mathbb{D}}u$, $i = 1, \ldots, j$.
Then $[u]$ and $\nu^1, \ldots, \nu^j$ satisfy

$$\int_0^T \langle [u], \varphi_t \rangle_{\mathscr{D}(\Omega)} + \langle f(\nu^1, \ldots, \nu^j), P_\partial^\dagger \varphi \rangle_{\mathscr{D}(\Omega)} dt + \\ + \int_\Omega u_0(x)\varphi(0, x) dx = 0.$$

for all $\varphi \in C^1([0, T], C^\infty(\Omega))$ with $\varphi(T, x) = 0$ for all $x \in \Omega$

The grid function formulation for the transport equation

The grid function formulation

$$\begin{cases} u_t = \mathbb{D}(u) \text{ in } \Omega_{\mathbb{X}} \\ u(0,x) = u_0(x). \end{cases}$$

The solution to this problem is $u(t) = e^{-t\mathbb{D}} u_0(x)$ for every (distributional, measure-valued, ...) initial data u_0.

Meaning of the solution

This is a generalization of the equality $e^{-tD} f(x) = f(x-t)$ for analytic functions.

The grid function formulation for the nonlinear diffusion

The grid function formulation

We discretize the Laplacian by using the grid derivative:

$$\Delta_{\mathbb{X}}{}^*\phi(u(x,t)) = \sum_{i=1}^{k} \mathbb{D}_i\mathbb{D}_i({}^*\phi(u(x-\varepsilon e_i, t))),$$

and impose Neumann boundary conditions by a first order finite-difference approximation.

We obtain the hyperfinite system of ODEs

$$\begin{cases} u_t = \Delta_{\mathbb{X}}{}^*\phi(u), & x \in \Omega_{\mathbb{X}},\ t \geq 0; \\ u\,(0,x) = {}^*u_0(x), & x \in \Omega_{\mathbb{X}}, \end{cases}$$

Properties of the grid solution

Existence and uniqueness of the grid solution (B. 2017)

The grid function formulation has a unique global solution $u : {}^*\mathbb{R}_{\geq 0} \to \Omega_{\mathbb{X}}$. Moreover, the mass of the solution is preserved:

$$\|u(t,x)\|_1 = \|{}^*u_0\|_1 \text{ for all } t \geq 0.$$

Coherence with the measure-valued solutions (B. 2017)

$[u]$ is a non-negative Radon measure, and $[{}^*\phi(u)] \in L^\infty(\Omega \times [0, T])$ for all $T \geq 0$. Moreover, it satisfies an entropy condition that characterizes admissible solutions to the original problem.

Conclusions

The strength of nonstandard analysis

None of this would be possible if the function * did not preserve first order properties!

Perspectives for future research

- Study classically ill-posed PDEs with grid functions (relevant also for many mathematical models of physical phenomena);
- develop a nonlinear theory of distributions;
- study the relations between grid functions, Colombeau's algebras, algebras of asymptotic functions, and spaces of ultrafunctions.

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Young measures

Definition (Young 1937, ...)

A Young measure over Ω is a measurable function $\nu : \Omega \to P(\mathbb{R})$. The "composition" between a function $g \in C_b^0(\mathbb{R})$ and ν is the function defined a.e. by

$$g(\nu)(x) = \int_{\mathbb{R}} g d\nu_x.$$

Theorem (Prokhorov 1956, Ball 1989, ...)

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions, uniformly bounded in $L^p(\Omega)$ for some $1 \leq p \leq \infty$. Then there exists a Young measure ν such that

$$\lim_{n\to\infty} \int_{\Omega} g(z_n(x))\varphi(x)dx = \int_{\Omega} \left(\int_{\mathbb{R}} g d\nu_x \right) \varphi(x)dx$$

for all $g \in C_b^0(\mathbb{R})$ and for all $\varphi \in L^1(\Omega)$.

Grid functions and Young measures

Theorem

For every $f \in \mathbb{G}(\Omega_{\mathbb{X}})$, *there exists a measurable* $\nu : \Omega \to \mathbb{M}(\mathbb{R})$ *such that*

$$^\circ\left(\sum_{x \in \Omega_{\mathbb{X}}} {}^*g(f(x))\ {}^*\varphi(x)\right) = \int_\Omega \left(\int_{\mathbb{R}} g d\nu_x\right) \varphi(x) dx$$

for all $g \in C^0_b(\mathbb{R})$ *and for all* $\varphi \in C^0(\mathbb{R})$. *Moreover, for all* $x \in \Omega$ *and for all Borel* $A \subseteq \mathbb{R}$, $0 \leq \nu_x(A) \leq 1$.

Proposition (Cutland 1986 & B. 2017)

For every $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ *such that* $\|f\|_\infty$ *is finite,* ν *is a Young measure and* $[f]$ *is its barycentre*