The Feferman-Vaught Theorem and products of finite fields

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Motivation

What can we say in the language $(0, 1, +, -, \cdot)$ about the direct product ring

$$\prod_p \mathbb{F}_p = \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5 \times \dots$$

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Motivation: model theory of the adeles, as studied by Angus Macintyre and Jamshid Derakhshan. $\prod_p \mathbb{F}_p$ arises as the "residue ring" of the valued ring of adeles over $\mathbb{Q}$.

Main Results

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Definable subsets of $\prod_p \mathbb{F}_p$ are Boolean combinations of $\exists\forall\exists$-definable sets.

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Proposition

In $\prod_p \mathbb{F}_p$, there is a definable copy of $(\mathcal{P}(\omega), \cap, \cup, -)$. In particular, $\prod_p \mathbb{F}_p$ has the independence property and the strict order property.

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We can vastly generalize support via the notion of "Boolean Valuation"

Interpreting Boolean Valuations

Definition (Boolean Valuation)

Let $\varphi(x_1, \ldots, x_n)$ be an L-formula. Let $a_1, \ldots, a_n \in \prod_p \mathbb{F}_p$. The *Boolean valuation of* $\varphi(\bar{x})$ *at* $\bar{a}$, denoted $\|\varphi(\bar{a})\|$, is the subset $\{p : \mathbb{F}_p \models \varphi(\bar{a}(p))\}$.

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For every formula $\varphi(\bar{x})$, the function sending $\bar{a}$ to the idempotent representing $\|\varphi(\bar{a})\|$ is $\emptyset$-definable in $\prod_p \mathbb{F}_p$.

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- $\exists$: Use the fact that $\|\exists z \varphi(z, \bar{a})\| = \bigcup_b \|\varphi(b, \bar{a})\|$.

The Feferman-Vaught Theorem

Notation

Let $\{M_\lambda : \lambda \in \Lambda\}$ be a family of L-structures. Let $M := \prod_{\lambda \in \Lambda} M_\lambda$ be the cartesian product structure.

Definition ($\|\varphi(\cdot)\|$)

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Theorem (Feferman-Vaught 1959)

Let $\phi(\bar{x})$ be a formula. Then there are L-formulas $\varphi_1(\bar{x}), \ldots, \varphi_m(\bar{x})$ and an $L_{Boolean}$ formula $\sigma(y_1, \ldots, y_m)$ such that for every $\bar{a} \in M$,

$$M \models \phi(\bar{a}) \textit{ iff } \mathcal{P}(\Lambda) \models \sigma(\|\varphi_1(\bar{a})\|, \ldots, \|\varphi_m(\bar{a})\|)$$

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Theorem (Q.E. for Complete Atomic Boolean Algebras (Tarski))

The theory of (the class of) complete atomic boolean algebras admits quantifier elimination when we add unary predicates $R_1, R_2, R_3, \ldots$ *where* $R_k(x)$ *signifies "x is above at least k atomic elements"*

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Corollary

Every definable subset of $M := \prod_{\lambda \in \Lambda} M_\lambda$ is a boolean combination of sets of the form $\{\bar{a} \in M : M_\lambda \models \varphi(\bar{a}(\lambda)) \text{ for at least } k \text{ indices } \lambda\}$, where $k \in \omega$ and $\varphi(\bar{x})$ is an L-formula.

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Corollary

Every definable subset of M is a boolean combination of open sets in this topology (in the language of algebraic geometry, every definable set is constructible)

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- Note that for any finitely given primes $p_1, \ldots, p_n$ and elements $a_1, \ldots, a_n$ with $a_k \in \mathbb{F}_{p_k}$, we can easily find elements $a, b \in \prod_p \mathbb{F}_p$ extending $\langle a_1, \ldots, a_n \rangle$ such that $a \in \bigoplus_p \mathbb{F}_p$ and $b \notin \bigoplus_p \mathbb{F}_p$.

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- Topologically, this means that both $\bigoplus_p \mathbb{F}_p$ and its complement are dense inside $\prod_p \mathbb{F}_p$ – that is, the boundary of $\bigoplus_p \mathbb{F}_p$ is all of $\prod_p \mathbb{F}_p$.
- However, it is easy to verify that the boundary of a boolean combination of open sets must be nowhere dense.
- Therefore, $\bigoplus_p \mathbb{F}_p$ is not definable inside $\prod_p \mathbb{F}_p$.
- This is also how we show that no $\mathbb{Z}a$ is definable (for non-torsion a):
- The set $\mathbb{Z}(1, 1, 1, \ldots)$ has this same dense-codense property.
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- However, the set $\{b : ba \in \mathbb{Z}a\}$ does; hence neither it nor $\mathbb{Z}a$ is definable.

Quantifier Reduction for $\prod_p \mathbb{F}_p$

Theorem (Feferman-Vaught (1959), "Tarski Reduction")

Every definable subset of $M := \prod_{\lambda \in \Lambda} M_\lambda$ is a boolean combination of sets of the form $\{\bar{a} \in M : M_\lambda \models \varphi(\bar{a}(\lambda))$ for at least k indices $\lambda\}$, where $k \in \omega$ and $\varphi(\bar{x})$ is an L-formula.

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Thank you!