Invertible binary algebras principally isotopic to a group

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Introduction

Auxiliary results

The setting of the problem

Main results
Definition 1
A binary groupoid $Q(A)$ is a non-empty set $Q$ together with a binary operation $A$. Binary groupoid $Q(A)$ is called quasigroup if for all ordered pairs $(a, b) \in Q^2$ exists unique solutions $x, y \in Q$ of the following equations:

$$A(a, x) = b, A(y, a) = b.$$ 

The solutions of these equations will be denoted by $x = A^{-1}(a, b)$ and $y = A^-(b, a)$, respectively.

Definition 2
A binary algebra $(Q; \Sigma)$ is called invertible algebra or system of quasigroups if each operation in $\Sigma$ is a quasigroup operation.
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Definition 2
A binary algebra $(Q; \Sigma)$ is called invertible algebra or system of quasigroups if each operation in $\Sigma$ is a quasigroup operation.
With each invertible algebra \((Q; \Sigma)\) the next five invertible algebras are connected:

\[(Q; \Sigma^{-1}), (Q; -\Sigma), (Q; -1(\Sigma^{-1})), (Q; (-\Sigma)^{-1}), (Q; \Sigma^*),\]

where

\[
\begin{align*}
\Sigma^{-1} &= \{ A^{-1} | A \in \Sigma \}, \\
-\Sigma &= \{ -1A | A \in \Sigma \}, \\
-1(\Sigma^{-1}) &= \{ -1(A^{-1}) | A \in \Sigma \}, \\
(-\Sigma)^{-1} &= \{ (-1A)^{-1} | A \in \Sigma \}, \\
\Sigma^* &= \{ A^* | A \in \Sigma \}.
\end{align*}
\]

Each of these invertible algebras are called parastrophies of the algebra \((Q; \Sigma)\).
Let us recall that the following absolutely closed second-order formula:

\[ \forall X_1, \ldots, X_m \forall x_1, \ldots, x_n \ (\omega_1 = \omega_2), \]
\[ \forall X_1, \ldots, X_k \exists X_{k+1} \ldots, X_m \forall x_1, \ldots, x_n \ (\omega_1 = \omega_2), \]

where \( \omega_1, \omega_2 \) are words written in the functional variables, \( X_1, \ldots, X_m \), and in the objective variables, \( x_1, \ldots, x_n \), are called \( \forall(\forall) \)-identity or hyperidentity and \( \forall\exists(\forall) \)-identity.

The satisfiability (truth) of these second order formula in the algebra \( (Q; \Sigma) \) is in the sense of functional quantifiers \( (\forall X_i) \) and \( (\exists X_j) \) meaning: ”for every value \( X_i = A \in \Sigma \) of the corresponding arity” and ”there exists a value \( X_j = A \in \Sigma \) of the corresponding arity”.

Definition 3
The groupoid \( Q(A) \) is called isotopic to the groupoid \( Q(B) \) if exist three maps \( \alpha, \beta, \gamma \) of \( Q \) to \( Q \) such that

\[
\gamma B(x, y) = A(\alpha x, \beta y)
\]

for all \( x, y \in Q \). The isotopy of the form \( T = (\alpha, \beta, \varepsilon) \), where \( \varepsilon \) is the identity map, is called principal isotope.

Definition 4
We say that a binary algebra \( (Q; \Sigma) \) is isotopic to the groupoid \( Q(\cdot) \), if each operation in \( \Sigma \) is isotopic to the groupoid \( Q(\cdot) \), i.e. for every operation \( A \in \Sigma \) there exists permutations \( \alpha_A, \beta_A, \gamma_A \) of \( Q \), that:

\[
\gamma_A A(x, y) = \alpha_A x \cdot \beta_A y,
\]

for every \( x, y \in Q \). Isotopy is called principal if \( \gamma_A = \varepsilon (\varepsilon - \text{unit permutation}) \) for every \( A \in \Sigma \).
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The groupoid $Q(A)$ is called isotopic to the groupoid $Q(B)$ if exist three maps $\alpha, \beta, \gamma$ of $Q$ to $Q$ such that

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In 1961 V.D. Belousov characterised quasigroups isotopic to groups and abelian groups.

**Theorem 5**

Let the nonempty set $Q$ form a quasigroup under four operations $A_i$ ($i=1,2,3,4$). If these operations satisfy the following identity:

$$A_1(A_2(x, y), z) = A_3(x, A_4(y, z)),$$

then there exists an operation ($\cdot$) under which $Q$ forms a group isotopic to all these four quasigroups.
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*then there exists an operation $\cdot$ under which $Q$ forms a group isotopic to all these four quasigroups.*
Theorem 6

Let the nonempty set $Q$ form a quasigroup under six operations $A_i$ ($i=1,2,3,4,5,6$). If these operations satisfy the following identity:

$$A_1(A_2(x, y), A_3(z, u)) = A_4(A_5(x, z), A_6(y, u)),$$

then there exists an operation $(\cdot)$ under which $Q$ forms an abelian group isotopic to all these six quasigroups, i.e.

$$A_1(x, y) = \alpha x \cdot \beta y, \quad A_4(x, y) = \chi x \cdot \varphi y,$$
$$A_2(x, y) = \alpha^{-1}(\gamma x \cdot \delta y), \quad A_5(x, y) = \chi^{-1}(\gamma x \cdot \theta y),$$
$$A_3(x, y) = \beta^{-1}(\theta x \cdot \psi y), \quad A_6(x, y) = \varphi^{-1}(\delta x \cdot \psi y),$$

where $\alpha, \beta, \gamma, \delta, \chi, \varphi, \psi, \theta$ are permutations of $Q$. 
We obtained characterizations of invertible algebras principally isotopic to a group or an abelian group by second-order formulas.
Theorem 7

The invertible algebra \((Q; \Sigma)\) is principally isotopic to a group, if and only if the following second-order formula

\[
A^{-1}A(B(x, B^{-1}(y, z)), u), v) = B(x, B^{-1}(y, A^{-1}A(z, u), v))),
\]

is valid in the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup^{-1} \Sigma)\) for all \(A, B \in \Sigma\).

Corollary 8

The class of quasigroups isotopic to groups is characterized by the following identity:

\[
x(y \langle (z/u)v)) = ((x(y \langle z))/u)v.
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\]

is valid in the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup^{-1} \Sigma)\) for all \(A, B \in \Sigma\).

Corollary 8
The class of quasigroups isotopic to groups is characterized by the following identity:

\[
x(y \backslash ((z/u)v)) = ((x(y \backslash z))/u)v.
\]
Theorem 9

The invertible algebra \((Q; \Sigma)\) is principally isotopic to an abelian group if and only if the following second-order formula:

\[
A^{-1} A(B(x, z), y), A^{-1}(u, B(w, y))) = \\
A^{-1} A(B(w, z), y), A^{-1}(u, B(x, y))).
\]

is valid in the algebra \((Q; \Sigma \cup \Sigma^{-1} \cup^{-1} \Sigma)\) for all \(A, B \in \Sigma\).

Corollary 10

The class of quasigroups isotopic to abelian groups is characterized by the following identity:

\[
((xz)/y)(u \setminus (wy)) = ((wz)/y)(u \setminus (xy)).
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Theorem 9
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The class of quasigroups isotopic to abelian groups is characterized by the following identity:

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The invertible algebra \((Q; \Sigma)\) with hyperidentity either (4.1), (4.2) or (4.3) is isotopic to an abelian group.
Invertible binary algebras principally isotopic to a group

Main results

Proposition

\[ X(Y(x, y), z) = Y(X(z, y), x) \] (4.1)

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The invertible algebra \((Q; \Sigma)\) with hyperidentity either (4.1), (4.2) or (4.3) is isotopic to an abelian group.