Invertible binary algebras principally isotopic to a group

Davit Shahnazaryan

Joint work with Sergey Davidov

Chair of Geometry and Algebra
Faculty of Mathematics and Mechanics
Yerevan State University

Logic Colloquium 2019

Definition 1

A binary groupoid $Q(A)$ is a non-empty set Q together with a binary operation A. Binary groupoid $Q(A)$ is called quasigroup if for all ordered pairs $(a, b) \in Q^2$ exists unique solutions $x, y \in Q$ of the following equations:

$$A(a, x) = b, A(y, a) = b.$$

The solutions of these equations will be denoted by $x = A^{-1}(a, b)$ and $y = {}^{-1}A(b, a)$, respectively.

Definition 2

A binary algebra $(Q; \Sigma)$ is called invertible algebra or system of quasigroups if each operation in Σ is a quasigroup operation.

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Definition 2

A binary algebra $(Q; \Sigma)$ is called invertible algebra or system of quasigroups if each operation in Σ is a quasigroup operation.

With each invertible algebra $(Q;\Sigma)$ the next five invertible algebras are connected:

$$(Q;\Sigma^{-1}),\ \ (Q;{}^{-1}\Sigma),\ \ (Q;{}^{-1}(\Sigma^{-1})),\ \ (Q;({}^{-1}\Sigma)^{-1}),\ \ (Q;\Sigma^{*}),$$

where

$$\begin{aligned}
\Sigma^{-1} &= \{A^{-1} \mid A \in \Sigma\},\\
{}^{-1}\Sigma &= \{{}^{-1}A \mid A \in \Sigma\},\\
{}^{-1}(\Sigma^{-1}) &= \{{}^{-1}(A^{-1}) \mid A \in \Sigma\},\\
({}^{-1}\Sigma)^{-1} &= \{({}^{-1}A)^{-1} \mid A \in \Sigma\},\\
\Sigma^{*} &= \{A^{*} \mid A \in \Sigma\}.
\end{aligned}$$

Each of these invertible algebras are called parastrophies of the algebra $(Q;\Sigma)$.

Let us recall that the following absolutely closed second-order formula:

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n \quad (\omega_1 = \omega_2),$$
$$\forall X_1, \ldots, X_k \exists X_{k+1} \ldots, X_m \forall x_1, \ldots, x_n \quad (\omega_1 = \omega_2),$$

where ω_1, ω_2 are words written in the functional variables, $X_1, \ldots, X_m$, and in the objective variables, $x_1, \ldots, x_n$, are called $\forall(\forall)$-identity or hyperidentity and $\forall\exists(\forall)$-identity.

The satisfiability (truth) of these second order formula in the algebra $(Q; \Sigma)$ is in the sense of functional quantifiers $(\forall X_i)$ and $(\exists X_j)$ meaning: "for every value $X_i = A \in \Sigma$ of the corresponding arity" and "there exists a value $X_j = A \in \Sigma$ of the corresponding arity".

Definition 3

The groupoid $Q(A)$ is called isotopic to the groupoid $Q(B)$ if exist three maps α, β, γ of Q to Q such that

$$\gamma B(x, y) = A(\alpha x, \beta y)$$

for all $x, y \in Q$. The isotopy of the form $T = (\alpha, \beta, \varepsilon)$, where ε is the identity map, is called principal isotope.

Definition 4

We say that a binary algebra $(Q; \Sigma)$ is isotopic to the groupoid $Q(\cdot)$, if each operation in Σ is isotopic to the groupoid $Q(\cdot)$, i.e. for every operation $A \in \Sigma$ there exists permutations $\alpha_A, \beta_A, \gamma_A$ of Q, that:

$$\gamma_A A(x, y) = \alpha_A x \cdot \beta_A y,$$

for every $x, y \in Q$. Isopoty is called principal if $\gamma_A = \epsilon(\epsilon$ - unit permutation) for every $A \in \Sigma$.

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In 1961 V.D. Belousov characterised quasigroups isotopic to groups and abelian groups.

Theorem 5

Let the nonempty set Q form a quasigroup under four operations A_i (i=1,2,3,4). If these operations satisfy the following identity:

$$A_1(A_2(x,y),z) = A_3(x, A_4(y,z)),$$

then there exists an operation $(\cdot)$ under which Q forms a group isotopic to all these four quasigroups.

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Theorem 6

Let the nonempty set Q form a quasigroup under six operations A_i (i=1,2,3,4,5,6). If these operations satisfy the following identity:

$$A_1(A_2(x,y),A_3(z,u)) = A_4(A_5(x,z),A_6(y,u)),$$

then there exists an operation $(\cdot)$ under which Q forms an abelian group isotopic to all these six quasigroups, i.e.

$$A_1(x,y) = \alpha x \cdot \beta y, \qquad A_4(x,y) = \chi x \cdot \varphi y,$$

$$A_2(x,y) = \alpha^{-1}(\gamma x \cdot \delta y), \qquad A_5(x,y) = \chi^{-1}(\gamma x \cdot \theta y),$$

$$A_3(x,y) = \beta^{-1}(\theta x \cdot \psi y), \qquad A_6(x,y) = \varphi^{-1}(\delta x \cdot \psi y),$$

where $\alpha, \beta, \gamma, \delta, \chi, \varphi, \psi, \theta$ are permutations of Q.

We obtained characterizations of invertible algebras principally isotopic to a group or an abelian group by second-order formulas.

Theorem 7

The invertible algebra $(Q;\Sigma)$ *is principally isotopic to a group, if and only if the following second-order formula*

$$A({}^{-1}A(B(x,B^{-1}(y,z)),u),v)=B(x,B^{-1}(y,A({}^{-1}A(z,u),v))),$$

is valid in the algebra $(Q;\Sigma\cup\Sigma^{-1}\cup{}^{-1}\Sigma)$ *for all* $A,B\in\Sigma$*.*

Corollary 8

The class of quasigroups isotopic to groups is characterized by the following identity:

$$x(y\backslash((z/u)v))=((x(y\backslash z))/u)v.$$

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The class of quasigroups isotopic to groups is characterized by the following identity:

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Theorem 9

The invertible algebra $(Q;\Sigma)$ is principally isotopic to an abelian group if and only if the following second-order formula:

$$A({}^{-1}A(B(x,z),y),A^{-1}(u,B(w,y))) =$$

$$= A({}^{-1}A(B(w,z),y),A^{-1}(u,B(x,y))).$$

is valid in the algebra $(Q;\Sigma\cup\Sigma^{-1}\cup{}^{-1}\Sigma)$ for all $A,B\in\Sigma$.

Corollary 10

The class of quasigroups isotopic to abelian groups is characterized by the following identity:

$$((xz)/y)(u\backslash(wy)) = ((wz)/y)(u\backslash(xy)).$$

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$$X(Y(x,y),z) = Y(X(z,y),x) \tag{4.1}$$

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