First-Order Model Theory of Free Projective Planes

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Inspiration: Tarski's Problems/Conjectures

In this talk our setting is first-order logic. The topic of this talk is the (first-order) model theory of free projective planes, but we spend few words explaining our inspiration.

Conjecture 1 (Tarski 1945)

Any two non-abelian free groups are elementary equivalent, and this common theory is decidable.

Sela (2006[1]) settled the first conjecture, and, independently, Kharlampovich and Myasnikov (2006[2]), settled both.

[1] Z. Sela. *Diophantine Geometry over Groups. VI. The Elementary Theory of a Free Group*. Geom. Funct. Anal. **16** (2006), no. 3, 707-730.

[2] O. Kharlampovich and A. Myasnikov. *Elementary Theory of Free Non-Abelian Groups*. J. Algebra **302** (2006), no. 2, 451-552.

Further Progress by Sela et al.

Sela (2013[3]) went deeper in the study of the model theory of free groups proving that this theory is (strictly[4]) stable.

Various other model theorists, e.g. Houcine, Perin, Pillay, Sklinos, and Tent, went on in this study proving many beautiful and deep results on the theory of free groups, e.g. characterization of elementary subgroups, homogeneity, characterization of the forking independence relation.

[3] Z. Sela. *Diophantine Geometry over Groups VIII: Stability*. Ann. of Math. (2) **177** (2013), no. 3, 787-868.

[4] The fact that this theory was not superstable was already known.

Free Objects in Model Theory

The model-theoretic analysis of free objects has been extended to many other structures of interest, e.g. free semigroups, free associative algebras, free monoids, and free Lie algebras.

In the present study we add to this picture a thorough study of the first-order theory of yet another classical notion of free object: Marshall Hall's free projective planes[5].

[5] M. Hall. *Projective Planes*. Trans. Amer. Math. Soc. **54** (1943), 229-277.

Projective Planes

Definition 2

A partial plane is a system of points and lines satisfying:

(A) there is at most one line through any two distinct points;

(B) there is at most one point through any two distinct lines.

We say that a partial plane is a projective plane if in (A)-(B) above we replace "at most" with "exactly one".

Convention 3

We say that a projective plane is non-degenerate if it contains four points such that no three of them are collinear. All the projective planes considered here will be non-degenerate!

Free Extensions

Definition 4

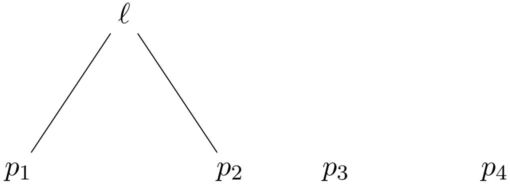
Given a partial plane P we can freely extend P to a projective plane $F(P)$ in the following way:

(1) if you two distinct points p_1 and p_2 are not joined by a line, then we add a new line $p_1 \vee p_2$ joining them;

(2) if you two distinct lines ℓ_1 and ℓ_2 are parallel, then we add a new point $\ell_1 \wedge \ell_2$ passing through them.

Repeat this ω many times and call the resulting plane $F(P)$.

In a Picture...



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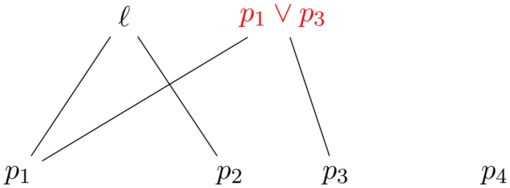

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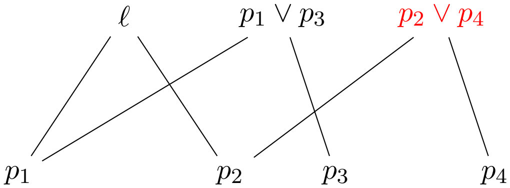

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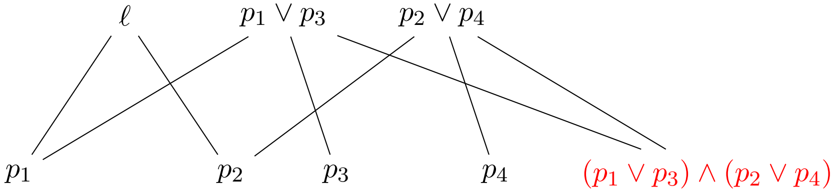


Free Projective Planes

Definition 5 (Hall)

Given $4 \leqslant n \leqslant \omega$, we let π_0^n be the partial plane consisting of a line ℓ, $n-2$ points on ℓ and 2 points off of ℓ. We let $\pi^n = F(\pi_0^n)$ (the free extension of π^n), and call it the free projective plane of rank n.

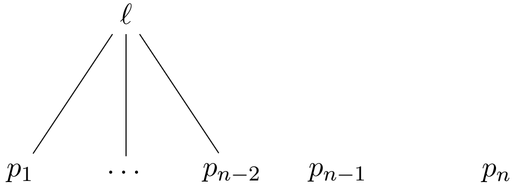


Figure: The partial plane π_0^n.

Some Facts on Free Projective Planes

A part from the pioneering studies of Hall, the free projective planes received the attention of eminent geometers such as Barlotti, Dembowski and Hughes, and of many other scholars. We mention here some structural results on these structures.

Fact 6 (Hall)

For $4 \leqslant n < m \leqslant \omega$, $\pi^n \not\cong \pi^m$.

Fact 7 (Hall + Kopeikina)

Subplanes (i.e. substructures which are projective plane in their own right) of free projective planes are free projective planes.

Open Projective Planes

Definition 8 (Hall)

Let P be a partial plane. We say that P is open if there is no finite subconfiguration A of P such that every point (resp. line) of A is incident with at least three lines (resp. points) of A.

Remark 9

Being open is a first-order property!

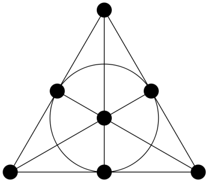


Figure: The Fano plane is not open.

Some Facts on Open Projective Planes

Fact 10 (Hall)

(1) *Free projective planes are open.*

(2) *Finitely generated open projective planes are free.*

Question 11 (Hall)

Does there exist an open projective plane which is not free?

Fact 12 (Kopeikina + Kelly)

There are $\aleph_0$-many countable non-free open projective planes.

Our Theorems

We will now state our main theorems. After this we will explain the techniques behind their proofs. The two main ingredients are the notion of open projective plane (already defined), and the notion of HF-constructibility (to be defined later). The second notion is essentially a notion of strong embedding, as in abstract elementary classes or Hrushovski's constructions.

Our results are model-theoretically inspired by the results on free groups mentioned above, but this is an entirely different area of mathematics and so the techniques are very different.

Choice of Language

Notation 13

We work in a language L with two sorts S_1 and S_2 specifying the set of points and the set of lines, and a symmetric binary relation I specifying the point-line incidence relation.

The Main Theorem

Not only we prove that the free projective planes are all elementary equivalent, but we also provide a canonical axiomatization of their theory. This is our main theorem.

Theorem 14 (Hyttinen and P.)

The theory T of open projective planes is complete.

Elementary Substructures

Furthermore, we give a complete characterization of the elementary substructure relation in models of the theory.

Theorem 15 (Hyttinen and P.)

If A and B are open projective planes and $A \subseteq B$, then A is elementary in B if and only if B is HF-constructible over A.

Stability

Theorem 16 (Hyttinen and P.)

The theory T of open projective planes is strictly stable, i.e. it is stable and not superstable.

Recall that also free groups are strictly stable.

Number of Models

Theorem 17 (Hyttinen and P.)

For every infinite cardinality κ there are 2^κ non-isomorphic open projective planes of power κ.

Clearly for uncountable κ's this follows from the unsuperstability of T, but for $\kappa = \aleph_0$ we need a separate proof.

Recall that previous to our work it was only known that there are $\aleph_0$-many countable non-free open projective planes.

The Main Corollary – Free Projective Planes

Corollary 18 (Hyttinen and P.)

The free projective planes $(\pi^n : 4 \leqslant n \leqslant \omega)$ *are all elementary equivalent, and they form an elementary chain with respect to the natural embeddings mapping* π_0^n *into* π_0^m*, for* $4 \leqslant n \leqslant m \leqslant \omega$*. Their common theory is the theory of open projective planes, and thus decidable and strictly stable.*

Forking in Open Projective Planes

Given partial planes $A \subseteq B$, we say that A is closed in B if A is closed under join of points and intersection of lines.

Definition 19

Given open partial planes A, B, C such that $B \cap C = A$ and A is a closed subplane of B and C, we say that $D \models T$ is a canonical amalgam of B and C over A, if D is well-foundedly F-constructible over its free relational amalgam $B \otimes_A C$.

Theorem 20

Let $\mathfrak{M}$ be the monster model of T, and $A, B, C \subseteq \mathfrak{M}$. Then $B \perp_A C$ (in the forking sense) if and only if $acl(ABC)$ is the canonical amalgam *of $acl(AB)$ and $acl(AC)$ over $acl(A)$.*

Type-Homogeneity

Theorem 21 (Hyttinen and P.)

The free projective plane π^ω *is strongly type-homogeneous, i.e. for every tuple* a, b *in* π^ω *and finite set of parameter* A *in* π^ω*,* a *and* b *have the same type over* A *if and only if there is* $f \in Aut(\pi^\omega)$ *mapping* a *to* b *and fixing* A *pointwise.*

In the case of free groups more is known: i.e. that every free group is type-homogeneous (as above with $A = \emptyset$) and that finitely generated free groups are strongly type-homogeneous.

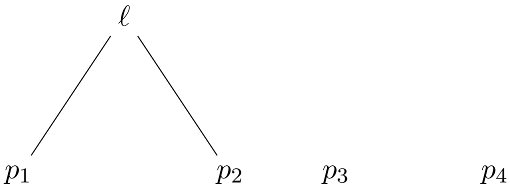
Well-founded HF-Constructions (Preliminaries)

We now introduce the main ingredients behind our proofs. We start by introducing the crucial concept of HF-construction.

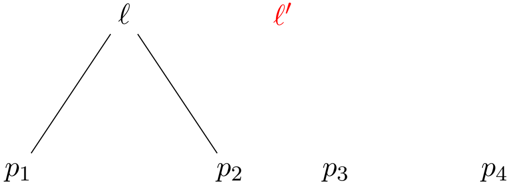
Definition 22

Let P be a partial plane and $P + x$ a partial plane containing P such that $x \notin P$ and $P + x = P \cup \{x\}$. We say that $P + x$ is a hyper-free (abbreviated as HF) one-point extension of P if x is incident with at most two elements of P. We say that $P + x$ is of type i, for $i = 0, 1, 2$, if in $P + x$ the element x is incident with exactly i elements of P. We denote this type as $t(P + x/P)$.

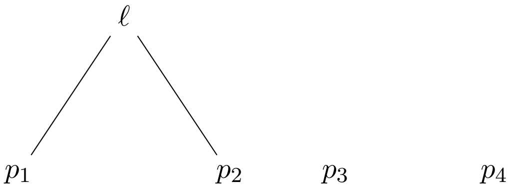
An Example of $t(P + x/P) = 0$



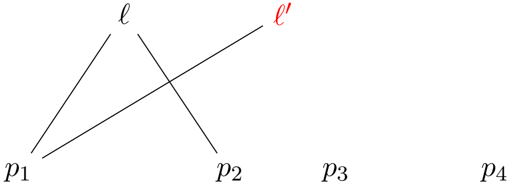
An Example of $t(P + x/P) = 0$



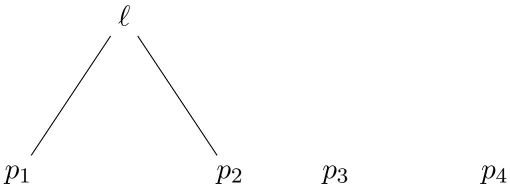
An Example of $t(P + x/P) = 1$



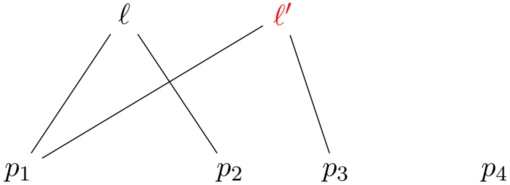
An Example of $t(P + x/P) = 1$



An Example of $t(P + x/P) = 2$



An Example of $t(P + x/P) = 2$



Well-founded HF-Constructions (Definition)

The following definition is due to Siebenmann[6].

Definition 23 (Siebenmann)

Let Q and P be countable partial planes. We say that P is well-foundedly HF-constructible from Q if there is a sequence $(P_k)_{k<\alpha\leqslant\omega}$ of partial planes such that:

(1) $P_0 = Q$;

(2) P_{k+1} is a hyper-free one-point extension of P_k;

(3) $\bigcup_{k<\alpha} P_k = P$.

We say in addition that P is F-constructible from Q if in the sequence $(P_k)_{k<\alpha\leqslant\omega}$ we have that $t(P_{k+1}/P_k) = 2$, for $k < \alpha$.

[6] L. C. Siebenmann. *A Characterization of Free Projective Planes*. Pacific J. Math. **15** (1965), 293-298.

Free Planes again

Remark 24

Clearly π^n $(4 \leqslant n \leqslant \omega)$ *is F-constructible from* π_0^n.

Remark 25

Clearly π^n $(4 \leqslant n \leqslant \omega)$ *is HF-constructible from* $\emptyset$ *(since the partial plane* π_0^n *is HF-constructible from* $\emptyset$*).*

In a Picture...

p_1

In a Picture...

p_1 p_2

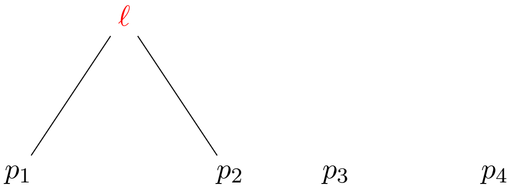
In a Picture...

p_1 p_2 p_3

In a Picture...

p_1 p_2 p_3 p_4

In a Picture...



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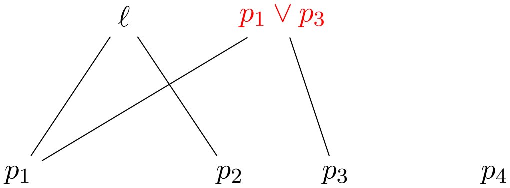

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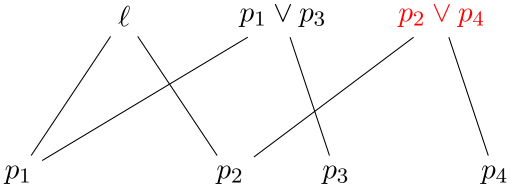

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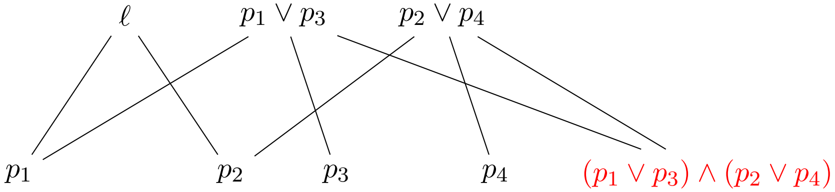

A Crucial Result of Siebenmann

On the other hand, in an arbitrary well-founded HF-construction at each stage we might add elements of type 0, or 1, or 2, and thus a-priori there are continuum many countable projective planes well-foundedly HF-constructible over $\emptyset$.

Fact 26 (Siebenmann)

The countable projective planes well-foundedly HF-constructible from $\emptyset$ are exactly the free projective planes π^n's ($4 \leqslant n \leqslant \omega$).

A Generalized Notion of HF-Construction

Definition 27 (Hyttinen and P.)

Let $A \subseteq B$ be partial planes (in particular A can be $\emptyset$), we say that B is HF-constructible (resp. F-constructible) from (or over) A if there is a linear ordering $(B - A, <)$ such that for every $b \in B - A$ there are at most two (resp. two) elements of B such that they are incident with b, and either from A or from $B - A$ and $<$-smaller than b. We call these HF-orderings.

Notice that HF-orderings need not be well-founded! For example, for every HF-ordering $<$ of a free projective plane A and ultraproduct A^* of A, we can extend naturally the order $<$ to an HF-ordering $<^*$ of A^*, but, unless the ultrafilter used to define A^* is principal, the HF-ordering $<^*$ is non-well-founded!

A Crucial Theorem

The notion of HF-construction is the real key behind a model-theoretic undestanding of free and open projective planes, and it is the most important tool in our proofs.

Theorem 28 (Hyttinen and P.)

Let P be a partial plane. Then P is open if and only if there is an HF-ordering (in the generalized sense) of P over $\emptyset$.

The left-to-right implication is the non-trivial one.

A Preliminary Fact

Fact 29 (Siebenmann)

Let A be a finite partial plane. Then A is open if and only if there is an HF-ordering of A over $\emptyset$ (nec. well-founded).

A Proof

Let P be an open partial plane and let $X_P = X$ be the set of all finite subconfigurations of P (substructures with respect to the choice of language). Then, by Fact 29, for every $A \in X$ we can find an HF-ordering $<_A$ of A over $\emptyset$ (since A is finite).

Let $\mathfrak{U}$ be an ultrafilter on X such that for all $A \in X$ we have:

$$X_A = \{B \in X : A \subseteq B\} \in \mathfrak{U}$$

(notice that the collection of sets of the form X_A have the finite intersection property, and so such an ultrafilter $\mathfrak{U}$ does exist).

A Proof (Cont.)

Now, for $A \in X$, let $<_A^1, ..., <_A^{n(A)}$ be an injective enumeration of the HF-orderings of A over $\emptyset$, and, for $0 < i \leqslant n(A)$, let:

$$Y_A^i = \{B \in X : A \subseteq B \text{ and } <_B \restriction A = <_i^A\}.$$

Notice that $X_A = Y_A^1 \cup \cdots \cup Y_A^{n(A)}$ and that for $0 < i < j \leqslant n(A)$ we have that $Y_A^i \cap Y_A^j = \emptyset$. Hence, being $\mathfrak{U}$ an ultrafilter, we can find a unique HF-ordering $<_A^*$ of A over $\emptyset$ such that $Y_A = \{B \in X : A \subseteq B \text{ and } <_B \restriction A = <_A^*\} \in \mathfrak{U}$.

A Proof (Cont.)

Notice now that for $A, B \in X$ such that $A \subseteq B$ we have that $<^*_A = <^*_B \restriction A$. In fact, since $Y_A, Y_B \in \mathfrak{U}$, we have that $Y_A \cap Y_B \neq \emptyset$. Let $C \in Y_A \cap Y_B$, then we have that $<^*_A = <_C \restriction A$ and $<^*_B = <_C \restriction B$, from which it follows $<^*_A = <^*_B \restriction A$. Thus, we can conclude that $<_* = \bigcup_{A \in X} <^*_A$ is an HF-ordering of P over $\emptyset$.

Directed Graphs

Definition 30

Given a HF-ordering of B over A we define a directed graph structure $(B, R_<)$ on B by letting $R_<(a, b) = R(a, b)$ if:

- $b \in B - A$;
- b is incident with a;
- either $a \in A$ or $a \in B - A$ and $a < b$.

The structure $(B, R_<)$ is in between B and $(B, <)$, and it encodes a lot of important information on B over A.

The corresponding (model-theoretic) expansion of the structure B (adding $R_<$) plays an important role in our proofs.

Three Levels of the Model-Theoretic Analysis

There are three levels in our model-theoretic analysis:

- Language of planes: $L_0 = (S_1, S_2, I)$;
- Language of planes with directed edges: $L_1 = (S_1, S_2, I, R)$;
- Language of planes with HF-orderings: $L_2 = (S_1, S_2, I, <)$.

Recap

We prove that the theory of free projective planes is complete and strictly stable and characterize the elementary substructure relation and the forking independence relation.

The main technical tool behind our proofs is the notion of HF-construction. The most interesting and difficult proofs are the proofs of completeness and unsuperstability.

Open Problems I

There are many other interesting questions on the theory T of open projective planes which are still open:

Open Problem 31

Does T have a prime model?

Notice that the theory of non-abelian free groups does not.

Open Problem 32

Describe the algebraic closure operator in models of T.

Open Problems II

Open Problem 33

Does T interpret an infinite field?

It has been propeved by Byron and Sklinos that no infinite field is definable in the theory of free groups.

Open Problem 34

Characterize the definable submodels in models of T.

New Perspectives

Notions of free and open object appear also in many other contexts in combinatorial geometry, most notably in the theory of n-gons – for a general study of these phenomena see [7].

We believe that behind our solutions to the main model theoretic questions concerning free projective planes there is a whole theory yet to be discovered, which focuses on non-well-founded constructions of combinatorial objects. We intend to return on this in an another more general future work.

[7] Martin Funk and Karl Strambach. *On Free Constructions*. Manuscripta Math. **72** (1991), no. 4, 335-374.

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