Generalised Miller Forcing May Collapse Cardinals

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Definition
A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called a $\kappa$-almost disjoint family if for $A \neq B \in \mathcal{A}$, $|A \cap B| < \kappa$. A $\kappa$-almost disjoint family of size at least $\kappa$ that is maximal is called a $\kappa$-mad family.

Observation
If $2^{<\kappa} = \kappa$, there is a $\kappa$-mad family $\mathcal{A} \subseteq [\kappa]^\kappa$ of size $2^\kappa$. 
The Forcing \(([\kappa]^{\kappa}, \subseteq)\)

Conditions are subsets of \(\kappa\) of size \(\kappa\). Stronger conditions are subsets. The separative quotient is \(([\kappa]^{\kappa}/ =^*, \subseteq^*)\).

Here, \(A \subseteq^* B\) if \(|A \setminus B| < \kappa\), and \(A =^* B\) if \(A \subseteq^* B\) and \(B \subseteq^* A\).

Observation

If \(([\kappa]^{\kappa}, \subseteq)\) collapses \(2^{\kappa}\) to \(\omega\), then there is a \(\kappa\)-mad family \(\mathcal{A}\) of size \(2^{\kappa}\).
The inverse direction

Theorem (Theorem 0.5 in [She07] Sh:861 from 2007)

(1) If there is a $\kappa$-ad subset of $[\kappa]^{\kappa}$ of size $\chi$, and if $\aleph_0 < \text{cf}(\kappa) = \kappa$ or if $\aleph_0 < \text{cf}(\kappa) < 2^{\text{cf}(\kappa)} \leq \kappa$, then the forcing $([\kappa]^{\kappa}, \subseteq)$ collapses $\chi$ to $\aleph_0$.

(2) Let $\kappa$ be uncountable. If there is a $\kappa$-ad subset of $[\kappa]^{\kappa}$ of size $\chi$, and of $\aleph_0 = \text{cf}(\kappa)$ then the forcing $([\kappa]^{\kappa}, \subseteq)$ collapses $\chi$ to $\aleph_1$. 
Definition

\( Q_\kappa \) is the following version of \( \kappa \)-Miller forcing: Conditions are trees \( T \subseteq {}^{\kappa^+} \kappa \) that are \( \kappa \) superperfect: for each \( s \in T \) there is \( s \subseteq t \) such that \( t \) is a \( \kappa \)-splitting node of \( T \) (short \( t \in \text{spl}(T) \)). A node \( t \in T \) is called a \( \kappa \)-splitting node if

\[
\text{osucc}_p(t) = \{ i < \kappa : t^\langle i \rangle \in T \}
\]

has size \( \kappa \). We furthermore require that the limit of an increasing in the tree order sequence of length less than \( \kappa \) of \( \kappa \)-splitting nodes is a \( \kappa \)-splitting node if it has length less than \( \kappa \).

For \( p, q \in Q_\kappa \) we write \( q \preceq_{Q_\kappa} p \) if \( q \subseteq p \). So subtrees are stronger conditions.
From \((\mathcal{P}(\kappa), \subseteq)\)-names to trees

**Lemma**

Suppose that \([\kappa]^\kappa\) collapses \(2^\kappa\) to \(\omega\). Then there is a \([\kappa]^\kappa\)-name \(\mathcal{I} : \mathbb{N}_0 \to 2^\kappa\) for a surjection, and there is a labelled tree \(\mathcal{T} = \{(a_\eta, n_\eta, \varrho_\eta) : \eta \in \omega > (2^\kappa)\}\) with the following properties

(a) \(a_\emptyset = \kappa\) and for any \(\eta \in \omega > (2^\kappa)\), \(a_\eta \in [\kappa]^\kappa\).

(b) \(\eta_1 \prec \eta_2\) implies \(a_{\eta_1} \supseteq a_{\eta_2}\).

(c) \(n_\eta \in [\lg(\eta) + 1, \omega)\).

(d) If \(a \in [\kappa]^\kappa\) then there is some \(\eta \in \omega > (2^\kappa)\) such that \(a \supseteq a_\eta\).

(e) If \(\eta^\langle \beta \rangle \in T\) then \(a_{\eta^\langle \beta \rangle}\) forces \(\mathcal{I} \upharpoonright n_\eta = \varrho_{\eta^\langle \beta \rangle}\) for some \(\varrho_{\eta^\langle \beta \rangle} \in n_\eta(2^\kappa)\), such that the \(\varrho_{\eta^\langle \beta \rangle}, \beta \in 2^\kappa\), are pairwise different. Hence for any \(\eta \in \omega > (2^\kappa)\), the family \(\{a_{\eta^\langle \alpha \rangle} : \alpha < 2^\kappa\}\) is a \(\kappa\)-ad family in \([a_\eta]^\kappa\).
Two types of long fusion sequences

Lemma

Let \( \langle \nu_\alpha : \alpha < \kappa^{\kappa} \rangle \) be an injective enumeration of \( \kappa^{\kappa} \) such that

\[
\nu_\alpha \triangleleft \nu_\beta \rightarrow \alpha < \beta.
\]

Let \( \langle p_\alpha, \nu_\alpha, c_\alpha : \alpha < \kappa^{\kappa} \rangle \) be a sequence such that for any \( \alpha \leq \lambda \) the following holds:

(a) \( p_0 \in \mathbb{Q}_\kappa \).

(b1) If \( \alpha = \beta + 1 < \kappa^{\kappa} \) and \( \nu_\beta \in sp(p_\beta) \), then

\[
c_\beta \in [\text{succ}_{p_\beta}(\nu_\beta)]^\kappa \text{ and } p_\alpha = p_\beta(\nu_\beta, c_\beta) := \bigcup \{ p_\beta^{\langle \nu_\beta^i \rangle} : i \in c_\beta \}
\]

\[
\bigcup \bigcup \{ p_\beta^{\langle \eta \rangle} : \eta \not\preceq \nu_\beta \land \nu_\beta \not\preceq \eta \}.
\]
Lemma

(b2) If $\alpha = \beta + 1 < \kappa^<\kappa$ and $\nu_\beta \notin \text{spl}(p_\beta)$ then $p_\alpha = p_\beta$.

(c) $p_\alpha = \bigcap \{ p_\beta : \beta < \alpha \}$ for limit $\alpha \leq \kappa^<\kappa$.

Then for any $\lambda \leq \kappa^<\kappa$, $p_\lambda \in Q^2_\kappa$ and $\forall \beta < \lambda, p_\beta \leq Q^2_\kappa p_\lambda$. 
By picture. Instead of choosing only $c_\beta \in \left[ \text{succ}_{p_\beta}(\nu_\beta) \right]^\kappa$ we choose for each $\nu_\beta \hat{i}$ one higher splitting point not necessarily the shortest one.

Why is the intersection still a Miller condition? At each splitting point in the sequence that stays, the successor set is shrunken at most once.
Definition

We assume $[\kappa]^\kappa$ collapses $2^\kappa$ to $\omega$. Let $\tau$ and $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho) : \eta \in \omega>(2^\kappa) \rangle$ be as in Lemma. Now let $Q_{\mathcal{T}}$ be the set of $Q_\kappa$-trees $p$ such that for every $\nu \in \text{spl}(p)$ there is $\eta_{p,\nu} = \eta_\nu \in \omega>(2^\kappa)$ such that

$$\text{osucc}_p(\nu) = \{ \varepsilon \in \kappa : \nu^\langle \varepsilon \rangle \in p \} = a_{\eta_\nu}.$$
Definition

We assume that $[\kappa]^\kappa$ collapses $2^\kappa$ to $\omega$ and the $\mathcal{T}$ is as above. For $T \in Q_\mathcal{T}$ and a splitting node $\nu$ of $T$ we set

$\varrho_{T,\nu} := \varrho_{\eta_{T,\nu}} \in \omega^{>(2^\kappa)}$. Recall $\eta_{T,\nu}$ is the translation of $\mathcal{T}$, and $\varrho$ is an initial segment of a collapsing function of $\mathcal{T}$. 
Definition

We assume that $[\kappa]^{\kappa}$ collapses $2^{\kappa}$ to $\omega$ and the $\mathcal{T}$ is as above. For $T \in \mathcal{Q}$ and a splitting node $\nu$ of $T$ we set

$$\varrho_{T,\nu} := \varrho_{\eta_{T,\nu}} \in \omega^{>}(2^{\kappa}).$$

Recall $\eta_{T,\nu}$ is the translation of $\mathcal{T}$, and $\varrho$ is an initial segment of a collapsing function of $\mathcal{T}$.

Definition

We assume that $[\kappa]^{\kappa}$ collapses $2^{\kappa}$ to $\omega$. Let $n \in \omega$.

$$D_n = \{ p \in \mathcal{Q} : (\forall \nu \in \text{spl}(p))(\lg(\varrho_{p,\nu}) > n) \}.$$
Lemma

We assume that $[\kappa]^\kappa$ collapses $2^\kappa$ to $\omega$, $\text{cf}(\kappa) > \omega$ and $2^{(\kappa<\kappa)} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa \rangle$ enumerate $\mathbb{Q}_\kappa$ such that each condition appears $2^\kappa$ times. There is $\langle (p_\alpha, n_\alpha, \bar{\gamma}_\alpha) : \alpha < 2^\kappa \rangle$ such that

(a) $n_\alpha < \omega$,

(b) $p_\alpha \in D_{n_\alpha}$ and $p_\alpha \geq T_\alpha$.

(c) If $\beta < \alpha$ and $n_\beta \geq n_\alpha$ then $p_\beta \perp p_\alpha$.

(d) $\bar{\gamma}_\alpha = \langle \gamma_{\alpha,\nu} : \nu \in \text{spl}(p_\alpha) \rangle$.

(e) $(\forall \nu \in \text{spl}(p_\alpha))(a_{\eta_{p_\alpha,\nu}} \models [\kappa]^\kappa \gamma_{\alpha,\nu} \in \text{range}(\rho_{p_\alpha,\nu}))$.

(f) $\gamma_{\alpha,\nu} \in 2^\kappa \setminus W_{<\alpha,\nu}$ with

$$W_{<\alpha,\nu} = \bigcup \{ \text{range}(\rho_{p_\beta,\nu}) : \beta < \alpha, \nu \in \text{spl}(p_\beta) \}.$$
Lemma

We assume that $[\kappa]^{\kappa}$ collapses $2^{\kappa}$ to $\omega$, $\text{cf}(\kappa) > \omega$ and $2^{(2^{<\kappa})} = 2^{\kappa}$. Let $\langle T_\alpha : \alpha < 2^{\kappa} \rangle$ enumerate all Miller trees that such each tree appears $2^{\kappa}$ times. If $\langle (p_\alpha, n_\alpha) : \alpha < 2^{\kappa} \rangle$ are such that

(a) $n_\alpha < \omega$,

(b) $p_\alpha \in D_{n_\alpha}$ and $p_\alpha \geq T_\alpha$,

(c) if $\beta < \alpha$ and $n_\beta = n_\alpha$ then $p_\beta \perp p_\alpha$,

(d) for any $k \in \omega$, $\{p_\alpha : n_\alpha \geq k\}$ is dense in $\mathcal{Q}_\kappa$.

Then there is a $\mathcal{Q}_\kappa$-name $\tau'$ for a surjection of $\omega$ onto $2^{\kappa}$. 

"Using less"
Definition
Let $B$ be a Boolean algebra. We write $B^+ = B \setminus \{0\}$. A subset $D \subseteq B^+$ is called dense if $(\forall b \in B^+)(\exists d \in D)(d \leq b)$.

Lemma
[Jec03, Lemma 26.7]. Let $(Q, <)$ be a notion of forcing such that $|Q| = \lambda > \aleph_0$ and such that $Q$ collapses $\lambda$ onto $\aleph_0$, i.e.,

$$0_Q \Vdash_Q |\check{\lambda}| = \aleph_0.$$

Then $\text{RO}(Q) = \text{Levy}(\aleph_0, \lambda)$. 
Lemma

If $[\kappa]^\kappa$ collapses $2^\kappa$ to $\aleph_0$, then $[\kappa]^\kappa$ is equivalent of $\text{Levy}(\aleph_0, 2^\kappa)$. $[\kappa]^\kappa$ has size $2^\kappa$. Hence Lemma 13 yields $\text{RO}([\kappa]^\kappa) = \text{Levy}(\aleph_0, 2^\kappa)$.

Proposition

If $[\kappa]^\kappa$ collapses $2^\kappa$ to $\aleph_0$, $\text{cf}(\kappa) > \aleph$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$ then $Q_{\kappa}$ is equivalent to $\text{Levy}(\aleph_0, 2^\kappa)$. 
Waiving conditions

Suppose that forcing with \([\kappa]^{\kappa}\) does not collapse \(2^{\kappa}\) (for regular \(\kappa\), this is equivalent to not having a \(\kappa\)-ad family of size \(2^{\kappa}\) in \([\kappa]^{\kappa}\).)

Or suppose that there is such a large ad family, but the density of our Miller forcing is \(> 2^{\kappa}\).

Then our proofs do not work.

Theorem (Theorem 5.4, 5.6, Baumgartner, Almost disjoint sets [Bau76])

Assume GCH in the ground model an force with

\[
P(\nu, \varrho) = \{ f : \varrho \to 2 : |\text{dom}(f)| < \nu \}
\]

ordered by extension. If \(\aleph_0 \leq \nu < \kappa = \text{cf}(\kappa)\) and \(\varrho \geq \kappa^{++}\), then in \(V[G]\), \(2^{\kappa} \geq \kappa^{++}\) and there is no \(\kappa\)-ad family in \([\kappa]^{\kappa}\) of size \(\kappa^{++}\).
Club $\kappa$-Miller forcing

Friedman, Zdomskyy [FZ10]. Brendle, Brooke-Taylor, Friedman, Montoya [BBTFM18]

Definition

Let $\kappa$ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Conditions in the forcing order $\mathbb{Q}_\kappa^{\text{club}}$ are trees $p \subseteq \kappa^{>\kappa}$ with the following additional properties:

(1) (Club filter superperfectness) For any $s \in p$ there is an extension $t \supseteq s$ in $p$ such that $\{\alpha \in \kappa : t\langle \alpha \rangle \in p\}$ is club in $\kappa$. We require that each node has either only one direct successor or splits into a club.

(2) (Closure of splitting) For each increasing sequence of length $< \kappa$ of splitting nodes, the union of the nodes on the sequence is a splitting node of $p$ as well.
More conditions

The forcing order is \( q \) is stronger than \( p \) iff \( q \subseteq p \).

We remark that clauses (1) and (2) imply:

(3) For every increasing sequence \( \langle t_i : i < \lambda \rangle \) of length \( \lambda < \kappa \) of nodes \( t_i \in p \in Q^\text{club}_\kappa \) we have that the limit of the sequence \( \bigcup \{ t_i : i < \lambda \} \) is also a node in \( p \).
Assume that \( \kappa^{<\kappa} \) is enumerated by \( \langle \eta_\alpha : \alpha < \kappa \rangle \).

**Definition**

We define \( \leq_\alpha \) slightly differently from Friedman and Zdomskyy [FZ10, Def. 2.2], so that the premise \( \kappa^{<\kappa} = \kappa \) suffices. For \( \alpha < \kappa \) we let

\[
\text{spl}_\alpha(p) = \{ t \in \text{spl}(p) : \text{otp}(\{ s \subsetneq t : s \in \text{spl}(p) \}) < \alpha \}
\]

and

\[
\text{cl}_\alpha(p) := \{ s \in p : \exists t \in \text{spl}_\alpha(p) s \subseteq t \land (\exists \beta < \alpha) (s = \eta_\beta) \}.
\]

We let \( p \leq_\alpha q \) if \( p \leq q \) and \( \text{cl}_\alpha(p) = \text{cl}_\alpha(q) \).
Note $|\text{cl}_\alpha(p)| \leq |\alpha| + \aleph_0 < \kappa$.

**Lemma**

*Then $(Q^\text{club}_\kappa, (\leq_\alpha)_{\alpha<\kappa})$ fulfils the fusion lemma.*

However, in iterations the diamond or Shelah’s $Dl$ is used in limit steps.
Preserving $\kappa^{++}$

**Definition**

Let $\mathbb{Q}$ be a forcing order and let $\lambda$ be a cardinal. $\text{Ax}(\mathbb{Q}, < \lambda)$ is the statement: For any set $D$ of size $< \lambda$ of dense sets in $\mathbb{Q}$ there is a filter $G \subseteq \mathbb{Q}$ such that $(\forall D \in D)(G \cap D \neq \emptyset)$.

**Theorem**

*Suppose that $\kappa > \omega$, $\kappa^{<\kappa} = \kappa$.*

1. $\text{Ax}(\mathbb{Q}_{\kappa}^{\text{club}}, < \kappa^{++})$ and $2^\kappa = \kappa^{++}$ is consistent relative to ZFC.

2. $\text{Ax}(\mathbb{Q}_{\kappa}^{\text{club}}, < \kappa^{++})$ implies that forcing with $\mathbb{Q}_{\kappa}^{\text{club}}$ does not collapse $\kappa^{++}$. 
A parallel Petr Simon’s result for Sacks

Theorem

Suppose

(a) $\kappa = \kappa^{<\kappa} > \omega$ and

(b) for every set $F \subseteq {}^{\kappa} \kappa$ of size $< 2^\kappa$ there is an eventually different $\kappa$ real $g$, i.e., an $g \in {}^\kappa \kappa$ such that

$$(\forall f \in F)(\exists \alpha_0 \in \kappa)(\forall \alpha \geq \alpha_0)(f(\alpha) \neq g(\alpha)).$$

Then $Q_{\kappa}^{\text{club}}$ and also Sacks forcing collapses $2^\kappa$ to $b_\kappa$. 
Definition

Let $\kappa < \lambda$ and let $\bar{\theta}$ be a sequence of ordinals. We write $\oplus_{\kappa, \lambda, \bar{\theta}}$ if the following holds:

(a) $\kappa$ is strongly inaccessible.

(b) $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \kappa \rangle$ is an increasing sequence of regular cardinals in $(2^{\|\varepsilon\|}, \kappa)$.

(c) $2^\kappa = \lambda$.

(d) $\text{tcf}(\prod_{\varepsilon < \kappa} \theta_\varepsilon, \leq_{\text{bd}}) = \lambda$. 
Theorem

(1) Assume that $\kappa$ is a strongly inaccessible cardinal, and that $\lambda = \lambda^\kappa = \text{cf}(\lambda)$. Then there is $\mathbb{P}$, a $(< \kappa)$-complete $\kappa^+$-cc notion of forcing such that in $\mathbb{P}$ forces: There is $\bar{\theta}$ with $\oplus_{\kappa,\lambda,\bar{\theta}}$.

(2) If $\oplus_{\kappa,\lambda,\bar{\theta}}$ then condition (b) of the previous Theorem holds the forcing $\mathbb{Q}_{\kappa}^{\text{club}}$ collapses $2^\kappa$ to $b_{\kappa} = \kappa^+$. 
The effect of $\kappa^{<\kappa}$

Theorem

If $\text{cf}(\kappa) = \kappa = \lambda^+$ and $\kappa \geq \theta^{++}$, and $\kappa^\theta > \kappa$, then $\mathbb{Q}_\kappa^{\text{club}}$ collapses $\kappa^\theta$ to $\kappa$.

Work is from preprints [MS18] [MS19]
By [Sh:351] for $\lambda^+ = \kappa$ there is a sequence $\bar{C}$ and there are $T, S_i, i < \lambda$ with the following properties:

1. $T = \{\alpha \in \kappa : \text{cf}(\alpha) \leq \theta\}$,
2. $T$ is the union of stationary sets $S_i, i < \lambda$, that have the following square property:
3. There is $\bar{C}^i = \langle C^i_\alpha : \alpha \in S_i \rangle$,
4. $C^i_\alpha$ is a closed subset of $\alpha$, not necessarily cofinal in $\alpha$, however, if $\alpha$ is a limit ordinal, then $C^i_\alpha$ is cofinal in $\alpha$, $C^i_\alpha \subseteq T \cap \alpha$ and $\text{otp}(C^i_\alpha) \leq \theta$,
5. for $\alpha \in S_i$, for any $\beta \in C^i_\alpha$, then $\beta \in S_i$ and $C^i_\beta = C^i_\alpha \cap \beta$. 


The last slide

Thank you!