Generalised Miller Forcing May Collapse Cardinals

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Antichains in $([\kappa]^\kappa, \subseteq)$

Definition

A family $\mathscr{A} \subseteq [\kappa]^\kappa$ is called a κ-almost disjoint family if for $A \neq B \in \mathscr{A}$, $|A \cap B| < \kappa$. A κ-almost disjoint family of size at least κ that is maximal is called a κ-mad family.

Observation

If $2^{<\kappa} = \kappa$, there is a κ-mad family $\mathscr{A} \subseteq [\kappa]^\kappa$ of size 2^κ.

The Forcing $([\kappa]^\kappa, \subseteq)$

Conditions are subsets of κ of size κ. Stronger conditions are subsets. The separative quotient is $([\kappa]^\kappa / =^*, \subseteq^*)$.
Here, $A \subseteq^* B$ if $|A \setminus B| < \kappa$, and
$A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$.

Observation

If $([\kappa]^\kappa, \subseteq)$ collapses 2^κ to ω, then there is a κ-mad family $\mathscr{A}$ of size 2^κ.

The inverse direction

Theorem (Theorem 0.5 in [She07] Sh:861 from 2007)

(1) If there is a κ-ad subset of $[\kappa]^\kappa$ of size χ, and if $\aleph_0 < \mathrm{cf}(\kappa) = \kappa$ or if $\aleph_0 < \mathrm{cf}(\kappa) < 2^{\mathrm{cf}(\kappa)} \leq \kappa$, then the forcing $([\kappa]^\kappa, \subseteq)$ collapses χ to $\aleph_0$.

(2) Let κ be uncountable. If there is a κ-ad subset of $[\kappa]^\kappa$ of size χ, and of $\aleph_0 = \mathrm{cf}(\kappa)$ then the forcing $([\kappa]^\kappa, \subseteq)$ collapses χ to $\aleph_1$.

Club in the tree order, but poor in the successors

Definition

$\mathbb{Q}_\kappa$ is the following version of κ-Miller forcing: Conditions are trees $T \subseteq {}^{\kappa>}\kappa$ that are κ *superperfect*: for each $s \in T$ there is $s \trianglelefteq t$ such that t is a κ-splitting node of T (short $t \in \mathrm{spl}(T)$). A node $t \in T$ is called a *κ-splitting node* if

$$\mathrm{osucc}_p(t) = \{i < \kappa \,:\, t\hat{}\langle i \rangle \in T\}$$

has size κ. We furthermore require that the limit of an increasing in the tree order sequence of length less than κ of κ-splitting nodes is a κ-splitting node if it has length less than κ.
For $p, q \in \mathbb{Q}_\kappa$ we write $q \leq_{\mathbb{Q}_\kappa} p$ if $q \subseteq p$. So subtrees are stronger conditions.

From $([\kappa]^\kappa, \subseteq)$-names to trees

Lemma

Suppose that $[\kappa]^\kappa$ collapses 2^κ to ω. Then there is a $[\kappa]^\kappa$-name $\underset{\sim}{\tau}\colon \aleph_0 \to 2^\kappa$ for a surjection, and there is a labelled tree $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho_\eta) \,:\, \eta \in {}^{\omega>}(2^\kappa)\rangle$ with the following properties

(a) $a_{\langle\rangle} = \kappa$ *and for any* $\eta \in {}^{\omega>}(2^\kappa)$, $a_\eta \in [\kappa]^\kappa$.

(b) $\eta_1 \triangleleft \eta_2$ *implies* $a_{\eta_1} \supseteq a_{\eta_2}$.

(c) $n_\eta \in [\lg(\eta) + 1, \omega)$.

(d) If $a \in [\kappa]^\kappa$ *then there is some* $\eta \in {}^{\omega>}(2^\kappa)$ *such that* $a \supseteq a_\eta$.

(e) If $\eta\hat{\ }\langle\beta\rangle \in T$ *then* $a_{\eta\hat{\ }\langle\beta\rangle}$ *forces* $\underset{\sim}{\tau} \restriction n_\eta = \varrho_{\eta\hat{\ }\langle\beta\rangle}$ *for some* $\varrho_{\eta\hat{\ }\langle\beta\rangle} \in {}^{n_\eta}(2^\kappa)$, *such that the* $\varrho_{\eta\hat{\ }\langle\beta\rangle}$, $\beta \in 2^\kappa$, *are pairwise different. Hence for any* $\eta \in {}^{\omega>}(2^\kappa)$, *the family* $\{a_{\eta\hat{\ }\langle\alpha\rangle} \,:\, \alpha < 2^\kappa\}$ *is a* κ*-ad family in* $[a_\eta]^\kappa$.

Two types of long fusion sequences

Lemma

Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa} \rangle$ *be an injective enumeration of* $\kappa^{<\kappa}$ *such that*

$$\nu_\alpha \lhd \nu_\beta \to \alpha < \beta.$$

Let $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha < \kappa^{<\kappa} \rangle$ *be a sequence such that for any* $\alpha \leq \lambda$ *the following holds:*

(a) $p_0 \in \mathbb{Q}_\kappa$.

(b1) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ *and* $\nu_\beta \in sp(p_\beta)$, *then*

$$c_\beta \in [\mathrm{succ}_{p_\beta}(\nu_\beta)]^\kappa \text{ and}$$

$$p_\alpha = p_\beta(\nu_\beta, c_\beta) := \bigcup \{ p_\beta^{\langle \nu_\beta \hat{} \langle i \rangle \rangle} : i \in c_\beta \} \cup \bigcup \{ p_\beta^{\langle \eta \rangle} : \eta \ntrianglelefteq \nu_\beta \wedge \nu_\beta \ntrianglelefteq \eta \}.$$

Lemma continued

Lemma

(b2) *If* $\alpha = \beta + 1 < \kappa^{<\kappa}$ *and* $\nu_\beta \notin \mathrm{spl}(p_\beta)$ *then* $p_\alpha = p_\beta$.

(c) $p_\alpha = \bigcap\{p_\beta : \beta < \alpha\}$ *for limit* $\alpha \leq \kappa^{<\kappa}$.

Then for any $\lambda \leq \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}^2_\kappa$ *and* $\forall \beta < \lambda$, $p_\beta \leq_{\mathbb{Q}^2_\kappa} p_\lambda$.

A slightly stronger descending fusion sequence

By picture. Instead of choosing only $c_\beta \in [\mathrm{succ}_{p_\beta}(\nu_\beta)]^\kappa$ we choose for each $\nu_\beta\hat{}i$ one higher splitting point not necessarily the shortest one.

Why is the intersection still a Miller condition? At each splitting point in the sequence that stays, the successor set is shrunken at most once.

τ and $\mathscr{T}$ in our Miller trees

Definition

We assume $[\kappa]^\kappa$ collapses 2^κ to ω. Let $\underset{\sim}{\tau}$ and $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho) : \eta \in {}^{\omega>}(2^\kappa) \rangle$ be as in Lemma. Now let $Q_{\mathscr{T}}$ be the set of $\mathbb{Q}_\kappa$-trees p such that for every $\nu \in \mathrm{spl}(p)$ there is $\eta_{p,\nu} = \eta_\nu \in {}^{\omega>}(2^\kappa)$ such that

$$\mathrm{osucc}_p(\nu) = \{\varepsilon \in \kappa : \nu\hat{\ }\langle \varepsilon \rangle \in p\} = a_{\eta_\nu}.$$

Translation

Definition

We assume that $[\kappa]^\kappa$ collapses 2^κ to ω and the $\mathcal{T}$ is as above. For $T \in Q_{\mathscr{T}}$ and a splitting node ν of T we set $\varrho_{T,\nu} := \varrho_{\eta_{T,\nu}} \in {}^{\omega>}(2^\kappa)$. Recall $\eta_{T,\nu}$ is the translation ot $\mathscr{T}$, and ϱ is an initial segment of a collapsing function of $\mathcal{T}$.

Translation

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Definition

We assume that $[\kappa]^\kappa$ collapses 2^κ to ω. Let $n \in \omega$.

$$D_n = \{p \in Q_{\mathscr{T}} \,:\, (\forall \nu \in \mathrm{spl}(p))(\lg(\varrho_{p,\nu}) > n)\}.$$

A $\mathbb{Q}_\kappa$-name for a collapse

Lemma

We assume that $[\kappa]^\kappa$ collapses 2^κ to ω, $\mathrm{cf}(\kappa) > \omega$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa \rangle$ enumerate $\mathbb{Q}_\kappa$ such that each condition appears 2^κ times. There is $\langle (p_\alpha, n_\alpha, \bar{\gamma}_\alpha) : \alpha < 2^\kappa \rangle$ such that

(a) $n_\alpha < \omega$,

(b) $p_\alpha \in D_{n_\alpha}$ *and* $p_\alpha \geq T_\alpha$.

(c) *If* $\beta < \alpha$ *and* $n_\beta \geq n_\alpha$ *then* $p_\beta \perp p_\alpha$.

(d) $\bar{\gamma}_\alpha = \langle \gamma_{\alpha,\nu} : \nu \in \mathrm{spl}(p_\alpha) \rangle$.

(e) $(\forall \nu \in \mathrm{spl}(p_\alpha))(a_{\eta_{p_\alpha,\nu}} \Vdash_{[\kappa]^{<\kappa}} \gamma_{\alpha,\nu} \in \mathrm{range}(\varrho_{p_\alpha,\nu}))$.

(f) $\gamma_{\alpha,\nu} \in 2^\kappa \setminus W_{<\alpha,\nu}$ *with*

$$W_{<\alpha,\nu} = \bigcup \{\mathrm{range}(\varrho_{p_\beta,\nu}) : \beta < \alpha, \nu \in \mathrm{spl}(p_\beta)\}.$$

Using less

Lemma

We assume that $[\kappa]^\kappa$ *collapses* 2^κ *to* ω, $\mathrm{cf}(\kappa) > \omega$ *and* $2^{(2^{<\kappa})} = 2^\kappa$. *Let* $\langle T_\alpha : \alpha < 2^\kappa \rangle$ *enumerate all Miller trees that such each tree appears* 2^κ *times. If* $\langle (p_\alpha, n_\alpha) : \alpha < 2^\kappa \rangle$ *are such that*

(a) $n_\alpha < \omega$,

(b) $p_\alpha \in D_{n_\alpha}$ *and* $p_\alpha \geq T_\alpha$,

(c) *if* $\beta < \alpha$ *and* $n_\beta = n_\alpha$ *then* $p_\beta \perp p_\alpha$,

(d) *for any* $k \in \omega$, $\{p_\alpha : n_\alpha \geq k\}$ *is dense in* $\mathbb{Q}_\kappa$.

Then there is a $\mathbb{Q}_\kappa$*-name* $\underset{\sim}{\tau}'$ *for a surjection of* ω *onto* 2^κ.

Characterising $\mathrm{RO}(\mathbb{P})$

Definition

Let B be a Boolean algebra. We write $B^+ = B \setminus \{0\}$. A subset $D \subseteq B^+$ is called *dense* if $(\forall b \in B^+)(\exists d \in D)(d \leq b)$.

Lemma

[Jec03, Lemma 26.7]. Let $(Q, <)$ *be a notion of forcing such that* $|Q| = \lambda > \aleph_0$ *and such that* Q *collapses* λ *onto* $\aleph_0$ *, i.e.,*

$$0_Q \Vdash_Q |\check{\lambda}| = \aleph_0.$$

Then $\mathrm{RO}(Q) = Levy(\aleph_0, \lambda)$*.*

Application to $([\kappa]^\kappa, \subseteq)$ and to $\mathbb{Q}_\kappa$

Lemma

If $[\kappa]^\kappa$ collapses 2^κ to $\aleph_0$, then $[\kappa]^\kappa$ is equivalent of $Levy(\aleph_0, 2^\kappa)$.

$[\kappa]^\kappa$ has size 2^κ. Hence Lemma 13 yields $\mathrm{RO}([\kappa]^\kappa) = \mathrm{Levy}(\aleph_0, 2^\kappa)$.

Proposition

If $[\kappa]^\kappa$ collapses 2^κ to $\aleph_0$, $\mathrm{cf}(\kappa) > \aleph$ and and $2^{(\kappa^{<\kappa})} = 2^\kappa$ then $\mathbb{Q}_\kappa$ is equivalent to $\mathrm{Levy}(\aleph_0, 2^\kappa)$.

Waiving conditions

Suppose that forcing with $[\kappa]^\kappa$ does not collapse 2^κ (for regular κ, this is equivalent to not having a κ-ad family of size 2^κ in $[\kappa]^\kappa$.)

Or suppose that there is such a large ad family, but the density of our Miller forcing is $> 2^\kappa$.

Then our proofs do not work.

Theorem (Theorem 5.4, 5.6, Baumgartner, Almost disjoint sets [Bau76])

Assume GCH in the ground model an force with

$$P(\nu, \varrho) = \{f\colon \varrho \to 2 \;:\; |\operatorname{dom}(f)| < \nu\}$$

ordered by extension. If $\aleph_0 \leq \nu < \kappa = \operatorname{cf}(\kappa)$ *and* $\varrho \geq \kappa^{++}$*, then in* $V[G]$*,* $2^\kappa \geq \kappa^{++}$ *and there is no* κ*-ad family in* $[\kappa]^\kappa$ *of size* κ^{++}*.*

Club κ-Miller forcing

Friedman, Zdomskyy [FZ10]. Brendle, Brooke-Taylor, Friedman, Montoya [BBTFM18]

Definition

Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Conditions in the forcing order $\mathbb{Q}_\kappa^{\mathrm{club}}$ are trees $p \subseteq {}^{\kappa>}\kappa$ with the following additional properties:

(1) (Club filter superperfectness) For any $s \in p$ there is an extension $t \trianglerighteq s$ in p such that $\{\alpha \in \kappa : t^\frown\langle\alpha\rangle \in p\}$ is club in κ. We require that each node has either only one direct successor or splits into a club.

(2) (Closure of splitting) For each increasing sequence of length $< \kappa$ of splitting nodes, the union of the nodes on the sequence is a splitting node of p as well.

More conditions

The forcing order is q is stronger than p iff $q \subseteq p$.

We remark that clauses (1) and (2) imply:

(3) For every increasing sequence $\langle t_i : i < \lambda \rangle$ of length $\lambda < \kappa$ of nodes $t_i \in p \in \mathbb{Q}_\kappa^{\mathrm{club}}$ we have that the limit of the sequence $\bigcup\{t_i : i < \lambda\}$ is also a node in p.

A version of $\leq_\alpha$

Assume that $\kappa^{<\kappa}$ is enumerated by $\langle \eta_\alpha : \alpha < \kappa \rangle$.

Definition

We define $\leq_\alpha$ slightly differently from Friedman and Zdomskyy [FZ10, Def. 2.2], so that the premise $\kappa^{<\kappa} = \kappa$ suffices.

For $\alpha < \kappa$ we let

$$\mathrm{spl}_\alpha(p) = \big\{ t \in \mathrm{spl}(p) \, : \, \mathrm{otp}(\{ s \subsetneq t \, : \, s \in \mathrm{spl}(p) \}) < \alpha \big\}$$

and

$$\mathrm{cl}_\alpha(p) := \{ s \in p \, : \, \exists t \in \mathrm{spl}_\alpha(p) s \subseteq t \wedge (\exists \beta < \alpha)(s = \eta_\beta) \}.$$

We let $p \leq_\alpha q$ if $p \leq q$ and $\mathrm{cl}_\alpha(p) = \mathrm{cl}_\alpha(q)$.

Fusion

Note $|\operatorname{cl}_\alpha(p)| \leq |\alpha| + \aleph_0 < \kappa$.

Lemma

Then $(\mathbb{Q}_\kappa^{\text{club}}, (\leq_\alpha)_{\alpha<\kappa})$ *fulfils the fusion lemma.*

However, in iterations the diamond or Shelah's Dl is used in limit steps.

Preserving κ^{++}

Definition

Let $\mathbb{Q}$ be a forcing order and let λ be a cardinal. $\mathrm{Ax}(\mathbb{Q}, <\lambda)$ is the statement For any set $\mathcal{D}$ of size $<\lambda$ of dense sets in $\mathbb{Q}$ there is a filter $G \subseteq \mathbb{Q}$ such that $(\forall D \in \mathcal{D})(G \cap D \neq \emptyset)$.

Theorem

Suppose that $\kappa > \omega$, $\kappa^{<\kappa} = \kappa$.

(1) $\mathrm{Ax}(\mathbb{Q}_\kappa^{\mathrm{club}}, <\kappa^{++})$ *and* $2^\kappa = \kappa^{++}$ *is consistent relative to* ZFC.

(2) $\mathrm{Ax}(\mathbb{Q}_\kappa^{\mathrm{club}}, <\kappa^{++})$ *implies that forcing with* $\mathbb{Q}_\kappa^{\mathrm{club}}$ *does not collapse* κ^{++}.

A parallel Petr Simon's result for Sacks

Theorem

Suppose

(a) $\kappa = \kappa^{<\kappa} > \omega$ *and*

(b) *for every set* $F \subseteq {}^{\kappa}\kappa$ *of size* $< 2^{\kappa}$ *there is an eventually different* κ *real* g, *i.e., an* $g \in {}^{\kappa}\kappa$ *such that* $(\forall f \in F)(\exists \alpha_0 \in \kappa)(\forall \alpha \geq \alpha_0)(f(\alpha) \neq g(\alpha))$.

Then $\mathbb{Q}_{\kappa}^{\mathrm{club}}$ *and also Sacks forcing collapses* 2^{κ} *to* $\mathfrak{b}_{\kappa}$.

A scale

Definition

Let $\kappa < \lambda$ and let $\bar{\theta}$ be a sequence of ordinals. We write $\oplus_{\kappa,\lambda,\bar{\theta}}$ if the following holds:

(a) κ is strongly inaccessible.

(b) $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \kappa \rangle$ is an increasing sequence of regular cardinals in $(2^{|\varepsilon|}, \kappa)$.

(c) $2^\kappa = \lambda$.

(d) $\mathrm{tcf}(\prod_{\varepsilon<\kappa} \theta_\varepsilon, \leq_{J_\kappa^{\mathrm{bd}}}) = \lambda$.

Getting eventually different reals

Theorem

(1) Assume that κ is a strongly inaccessible cardinal, and that $\lambda = \lambda^\kappa = \mathrm{cf}(\lambda)$. Then there is $\mathbb{P}$, a $(< \kappa)$-complete κ^+-cc notion of forcing such that in $\mathbb{P}$ forces: There is $\bar{\theta}$ with $\oplus_{\kappa,\lambda,\bar{\theta}}$.

(2) If $\oplus_{\kappa,\lambda,\bar{\theta}}$ then condition (b) of the previous Theorem holds the forcing $\mathbb{Q}_\kappa^{\mathrm{club}}$ collapses 2^κ to $\mathfrak{b}_\kappa = \kappa^+$.

The effect of $\kappa^{<\kappa}$

Theorem

If $\mathrm{cf}(\kappa) = \kappa = \lambda^+$ *and* $\kappa \geq \theta^{++}$*, and* $\kappa^\theta > \kappa$*, then* $\mathbb{Q}_\kappa^{\mathrm{club}}$ *collapses* κ^θ *to* κ*.*

Work is from preprints [MS18] [MS19]

Guessing devices in ZFC

By [Sh:351] for $\lambda^+ = \kappa$ there is a sequence $\bar{C}$ and there are T, S_i, $i < \lambda$ with the following properties:

(1) $T = \{\alpha \in \kappa : \mathrm{cf}(\alpha) \leq \theta\}$,

(2) T is the union of stationary sets S_i, $i < \lambda$, that have the following square property:

(3) There is $\bar{C}^i = \langle C^i_\alpha : \alpha \in S_i \rangle$,

(4) C^i_α is a closed subset of α, not necessarily cofinal in α, however, if α is a limit ordinal, then C^i_α is cofinal in α, $C^i_\alpha \subseteq T \cap \alpha$ and $\mathrm{otp}(C^i_\alpha) \leq \theta$,

(5) for $\alpha \in S_i$, for any $\beta \in C^i_\alpha$, then $\beta \in S_i$ and $C^i_\beta = C^i_\alpha \cap \beta$.

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The last slide

Thank you!