Computability theory, reverse mathematics, and Hindman’s Theorem

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HT is a $\Pi^1_2$ principle, of the form

$$\forall X \left[ \Phi(X) \rightarrow \exists Y \psi(X, Y) \right]$$

with $\Phi$ and $\psi$ arithmetic.
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with \( \Phi \) and \( \psi \) arithmetic.

We can think of such a principle as a **problem**.

An **instance** of such a problem is an \( X \) s.t. \( \Phi(X) \) holds.

A **solution** to this instance is a \( Y \) s.t. \( \psi(X, Y) \) holds.
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This is a natural context for computability-theoretic analysis.
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\( \text{ACA}_0 \) corresponds roughly to arithmetic mathematics.

\( \text{ACA}_0 \) proves that for every \( X \), the jump \( X' \) exists, and hence that so does each finite iterate \( X^{(n)} \).
We can also employ the perspective of reverse mathematics:

**RCA**₀ is the usual weak base system of reverse mathematics, corresponding roughly to computable mathematics.

All implications below are over RCA₀.

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ACA₀ proves that for every X, the jump X’ exists, and hence that so does each finite iterate X^(n).

**ACA**₀⁺ adds to ACA₀ that for every X, the ωˢᵗ jump X^(ω) exists.
Thm (Blass, Hirst, and Simpson).

1. Every computable instance of HT has an $\emptyset^{(\omega+1)}$-computable solution.

2. There is a computable instance of HT all of whose solutions compute $\emptyset'$.

3. $\text{ACA}_0^+ \rightarrow \text{HT}$. 

4. $\text{HT} \rightarrow \text{ACA}_0$. 
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Open Question. Does HT hold arithmetically? Does $\text{ACA}_0 \rightarrow \text{HT}$?
Iterated Hindman’s Theorem (IHT): For instances $c_0, c_1, \ldots$ of HT, there are $x_0 < x_1 < \cdots$ s.t. each $\{x_i : i \geq n\}$ is a solution to $c_n$. The results of Blass, Hirst, and Simpson also hold for IHT. One way to prove (I)HT is to use idempotent ultrafilters. Let $A_k = \{n : n + k \in A\}$. The set of ultrafilters on $\mathbb{N}$ is a semigroup under the operation $U \oplus V = \{A : \{k : A_k \in U\} \in V\}$. $U$ is idempotent if $U \oplus U = U$. Hirst showed that IHT is equivalent to the existence of certain countable approximations to idempotent ultrafilters.
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Montalbán and Shore developed a framework to expand the language of second-order arithmetic to talk about ultrafilters. They showed that ACA$_0$ plus the existence of an idempotent ultrafilter implies IHT. They also showed that the existence of an idempotent ultrafilter is conservative over ACA$_0$ plus IHT, ACA$_0^+$, and several other systems. Kreuzer also showed the $\Pi^1_2$-conservativity of the existence of an idempotent ultrafilter over ACA$_0$ plus IHT and ACA$_0^+$ by working in higher-order reverse mathematics.

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**Thm (Carlucci, Kołodzieczyk, Lepore, and Zdanowski)**. \( \text{HT}^{\leq 2} \rightarrow \text{ACA}_0 \). 

$HT^n$ is HT for sums of exactly $n$ many elements.
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$[X]^n$ is the set of $n$-element subsets of $X$.

$\text{RT}^n$: For every coloring of $[\mathbb{N}]^n$ with finitely many colors, there is an infinite $H \subseteq \mathbb{N}$ s.t. every element of $[H]^n$ has the same color.

It is easy to see that $\text{RT}^n \rightarrow \text{HT}^=n$. 
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**Thm (Seetapun).** RT$^2 \rightarrow$ ACA$_0$.

So HT$^=2$ is strictly weaker than ACA$_0$. 
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**Thm (Seetapun).** $RT^2 \rightarrow ACA_0$.

So $HT^=2$ is strictly weaker than $ACA_0$.

**Question (Dzhafarov, Jockusch, Solomon, and Westrick).** Is $HT^=2$ computably true? Is it provable in $RCA_0$?
Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). \( HT^{=2} \) is not computably true, and hence is not provable in \( \text{RCA}_0 \).
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Building a computable instance $c : \mathbb{N} \rightarrow 2$ of $\text{HT}^2$ with no computable solution:
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Building a computable instance \( c : \mathbb{N} \rightarrow 2 \) of \( HT^{=2} \) with no computable solution:

Let \( X + s = \{ k + s : k \in X \} \) and let \( W_i \) be the \( i^{\text{th}} \) c.e. set.
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Building a computable instance $c : \mathbb{N} \to 2$ of $HT^=2$ with no computable solution:

Let $X + s = \{k + s : k \in X\}$ and let $W_i$ be the $i^{th}$ c.e. set.

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Ensure that $F_i + s$ is not monochromatic for all sufficiently large $s$. 
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Problem: interactions between the strategies for different $i$'s.
Think of the $c(n)$’s as mutually independent random variables, with values 0 and 1 each having probability $\frac{1}{2}$. 
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These events for $F_i + s$ and $F_j + t$ are independent when $s$ and $t$ are sufficiently far apart.
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So we need to know that when events with “sufficiently small” probability are “sufficiently independent” then it is possible to avoid them all.
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So we need to know that when events with “sufficiently small” probability are “sufficiently independent” then it is possible to avoid them all effectively.
To do this, we use the **Computable Lovász Local Lemma of Rumyantsev and Shen**, in the form of the following corollary:
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For each $q \in (0, 1)$ there is an $M$ s.t. the following holds.

Let $E_0, E_1, \ldots$ be a computable sequence of finite sets, each of size at least $M$.

Suppose that for each $m \geq M$ and $n$, there are at most $2^{qm}$ many $i$ s.t. $|E_i| = m$ and $n \in E_i$, and that we can compute the set of all such $i$ given $m$ and $n$.

Then there is a computable $c : \mathbb{N} \to 2$ s.t. for each $i$ the set $E_i$ is not monochromatic for $c$. 
Building a computable instance $c : \mathbb{N} \to 2$ of $\text{HT}^2$ with no computable solution:

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Use the computable LLL to ensure that $F_i + s$ is not monochromatic for all sufficiently large $s$. 

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). There is a computable instance of $\text{HT}^2$ s.t. every solution is diagonally noncomputable (DNC) relative to $\emptyset'$. 
Building a computable instance $c : \mathbb{N} \to 2$ of HT$^=2$ with no computable solution:

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Use the computable LLL to ensure that $F_i + s$ is not monochromatic for all sufficiently large $s$.

We can work with $W_i^{\emptyset'}$ instead, to obtain a $c$ with to $\Sigma^0_2$ solution.
Building a computable instance \( c : \mathbb{N} \to 2 \) of \( \text{HT}^=2 \) with no computable solution:

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Use the computable LLL to ensure that \( F_i + s \) is not monochromatic for all sufficiently large \( s \).

We can work with \( W_i^{\emptyset'} \) instead, to obtain a \( c \) with to \( \Sigma^0_2 \) solution.

The sizes of the \( F_i \) can be computably bounded, so we can also ensure that solutions to \( c \) are effectively immune relative to \( \emptyset' \).
Building a computable instance $c : \mathbb{N} \rightarrow 2$ of HT$=^2$ with no computable solution:

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Use the computable LLL to ensure that $F_i + s$ is not monochromatic for all sufficiently large $s$.

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**Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick).** There is a computable instance of HT$=^2$ s.t. every solution is diagonally noncomputable (DNC) relative to $\emptyset'$. 
2-DNC is the reverse-mathematical principle corresponding to diagonal noncomputability relative to the jump.
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\[ \text{RRT}_2^2: \text{ If } c : [\mathbb{N}]^2 \to \mathbb{N} \text{ is s.t. } |c^{-1}(i)| \leq 2 \text{ for all } i, \text{ there is an infinite } R \subseteq \mathbb{N} \text{ s.t. } c \text{ is injective on } [R]^2. \]

Thm (J. Miller). \( \text{RRT}_2^2 \leftrightarrow 2\text{-DNC}. \)
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\textbf{Thm (J. Miller).} \text{ RRT}_2^2 \leftrightarrow 2\text{-DNC.}

\textbf{Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick).} \text{ HT}^=^2 \to 2\text{-DNC.}

\textbf{Open Question.} \text{ Does } 2\text{-DNC } \to \text{ HT}^=^2? \\
\textbf{Open Question.} \text{ Does } \text{ HT}^=^2 \to \text{ RT}^2?
An instance of HT$^{< 2}$ or RT$^2$ might have no solution containing a given $n$. 

HT$^2$ has a solution containing $n$.
An instance of $\text{HT}^{\leq 2}$ or $\text{RT}^2$ might have no solution containing a given $n$.

However, every instance of $\text{HT}^2$ does have such a solution:

$\text{HT}^2(n)$ is basically $\text{HT}^{\leq 2}$.
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HT\(^2\)(0) is basically HT\(\leq 2\).

We can pass between HT\(^2\)(0) and HT\(^2\)(n) by translating the coloring by 2\(n\) and then translating the solution back by \(n\).

Thus every HT\(^2\)(n) is equivalent to HT\(\leq 2\).
HT is equivalent to the **Finite Union Theorem (FUT)**: For every coloring of the finite subsets of $\mathbb{N}$ with finitely many colors, there are nonempty finite sets $F_0 < F_1 < F_2 < \cdots$ such that all nonempty finite unions of the $F_i$'s have the same color.
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Hirst considered the following variation, motivated by a lemma of Hilbert:

**HIL**: For every coloring of the finite subsets of $\mathbb{N}$ with finitely many colors, there are distinct nonempty finite sets $F_0, F_1, F_2, \ldots$ such that all nonempty finite unions of the $F_i$’s have the same color.
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**Thm (Hirst)**. $\text{HIL} \iff \text{RT}^1$.

Thus HIL is computably true (though not quite provable in $\text{RCA}_0$).
Let $P$ be a version of HT.

$P_k$ is $P$ restricted to $k$-colorings.
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Let $\lambda(n)$ be the least exponent of $n$ base 2, and let $\mu(n)$ be the greatest exponent of $n$ base 2.

$S \subseteq \mathbb{N}$ satisfying apartness if for all $m < n$ in $S$, we have $\mu(m) < \lambda(n)$.

P with apartness is $P$ with the extra condition that the solution satisfy apartness.
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$P$ with **apartness** is $P$ with the extra condition that the solution satisfy apartness.

Thinking of HT as FUT makes apartness natural.

$HT_k$ and $HT_k$ with apartness are equivalent to $FUT_k$ and hence to each other.
Thm (Carlucci, Kołodzieczyk, Lepore, and Zdanowski).

1. $\text{HT}^\leq_n$ with apartness is equivalent to $\text{FUT}^\leq_n$, and also for $=n$.

2. $\text{HT}^\leq_2$ implies $\text{HT}^\leq_n$ with apartness.

3. $\text{HT}^\leq_2$ with apartness implies $\text{ACA}_0$.

4. $\text{HT}^\leq_4$ implies $\text{ACA}_0$.

5. For $n \geq 3$, $\text{HT}^\leq_n$ with apartness is equivalent to $\text{ACA}_0$.
Thm (Carlucci, Kołodzieczyk, Lepore, and Zdanowski).

1. $\text{HT}^{\leq n}_k$ with apartness is equivalent to $\text{FUT}^{\leq n}_k$, and also for $=n$.

2. $\text{HT}^k_{2k}$ implies $\text{HT}^k_{n}$ with apartness.
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1. $\text{HT}_{k}^{\leq n}$ with apartness is equivalent to $\text{FUT}_{k}^{\leq n}$, and also for $=n$.

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3. $\text{HT}_{2}^{\leq 2}$ with apartness implies $\text{ACA}_0$.

4. $\text{HT}_{4}^{\leq 2}$ implies $\text{ACA}_0$. 

Thm (Dzhafarov, Jockusch, Solomon, and Westrick).

1. $\text{HT}_{2}^{\leq 3}$ implies $\text{ACA}_0$.

2. $\text{HT}_{2}^{\leq 2}$ implies the stable version of $\text{RT}_{2}^{\leq 2}$ over $B_{\Sigma^0_2}$.
Thm (Carlucci, Kołodzieczyk, Lepore, and Zdanowski).

1. $HT_{k}^{\leq n}$ with apartness is equivalent to $FUT_{k}^{\leq n}$, and also for $\equiv n$.

2. $HT_{2k}^{\leq n}$ implies $HT_{k}^{\leq n}$ with apartness.

3. $HT_{2}^{\leq 2}$ with apartness implies $ACA_0$.

4. $HT_{4}^{\leq 2}$ implies $ACA_0$.

5. For $n \geq 3$, $HT_{k}^{\equiv n}$ with apartness is equivalent to $ACA_0$. 
Thm (Carlucci, Kołodziejczyk, Lepore, and Zdanowski).

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Thm (Dzhafarov, Jockusch, Solomon, and Westrick).

1. $\text{HT}_{3}^{\leq 3}$ implies $\text{ACA}_0$.

2. $\text{HT}_{2}^{\leq 2}$ implies the stable version of $\text{RT}_2^2$ over $\text{BSigma}_2^0$. 