Computability theory, reverse mathematics, and Hindman's Theorem

Denis R. Hirschfeldt

University of Chicago

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

HT is a Π^1_2 principle, of the form

$$\forall X \, [\Phi(X) \rightarrow \exists Y \, \Psi(X, Y)]$$

with Φ and Ψ arithmetic.

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

HT is a Π^1_2 principle, of the form

$$\forall X\,[\Phi(X) \ \rightarrow\ \exists Y\,\Psi(X, Y)]$$

with Φ and Ψ arithmetic.

We can think of such a principle as a **problem**.

An **instance** of such a problem is an X s.t. $\Phi(X)$ holds.

A **solution** to this instance is a Y s.t. $\Psi(X, Y)$ holds.

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

HT is a Π^1_2 principle, of the form

$$\forall X\,[\Phi(X) \;\rightarrow\; \exists Y\,\Psi(X, Y)]$$

with Φ and Ψ arithmetic.

We can think of such a principle as a **problem**.

An **instance** of such a problem is an X s.t. $\Phi(X)$ holds.

A **solution** to this instance is a Y s.t. $\Psi(X, Y)$ holds.

This is a natural context for computability-theoretic analysis.

We can also employ the perspective of reverse mathematics:

We can also employ the perspective of reverse mathematics:

$\mathbf{RCA_0}$ is the usual weak base system of reverse mathematics, corresponding roughly to computable mathematics.

All implications below are over RCA_0.

We can also employ the perspective of reverse mathematics:

$\mathbf{RCA_0}$ is the usual weak base system of reverse mathematics, corresponding roughly to computable mathematics.

All implications below are over RCA_0.

$\mathbf{ACA_0}$ corresponds roughly to arithmetic mathematics.

ACA_0 proves that for every X, the jump X' exists, and hence that so does each finite iterate $X^{(n)}$.

We can also employ the perspective of reverse mathematics:

RCA_0 is the usual weak base system of reverse mathematics, corresponding roughly to computable mathematics.

All implications below are over RCA_0.

ACA_0 corresponds roughly to arithmetic mathematics.

ACA_0 proves that for every X, the jump X' exists, and hence that so does each finite iterate $X^{(n)}$.

ACA_0^+ adds to ACA_0 that for every X, the ω^{th} jump $X^{(\omega)}$ exists.

Thm (Blass, Hirst, and Simpson).

1. Every computable instance of HT has an $\emptyset^{(\omega+1)}$-computable solution.

2. There is a computable instance of HT all of whose solutions compute $\emptyset'$.

3. $\mathsf{ACA}_0^+ \to \mathsf{HT}$.

4. $\mathsf{HT} \to \mathsf{ACA}_0$.

Thm (Blass, Hirst, and Simpson).

1. Every computable instance of HT has an $\emptyset^{(\omega+1)}$-computable solution.

2. There is a computable instance of HT all of whose solutions compute $\emptyset'$.

3. $\mathsf{ACA}_0^+ \rightarrow \mathsf{HT}$.

4. $\mathsf{HT} \rightarrow \mathsf{ACA}_0$.

Open Question. Does HT hold arithmetically? Does $\mathsf{ACA}_0 \rightarrow \mathsf{HT}$?

Iterated Hindman's Theorem (IHT): For instances $c_0, c_1, \ldots$ of HT, there are $x_0 < x_1 < \cdots$ s.t. each $\{x_i : i \geqslant n\}$ is a solution to c_n.

Iterated Hindman's Theorem (IHT): For instances $c_0, c_1, \ldots$ of HT, there are $x_0 < x_1 < \cdots$ s.t. each $\{x_i : i \geqslant n\}$ is a solution to c_n.

The results of Blass, Hirst, and Simpson also hold for IHT.

Iterated Hindman's Theorem (IHT): For instances $c_0, c_1, \ldots$ of HT, there are $x_0 < x_1 < \cdots$ s.t. each $\{x_i : i \geqslant n\}$ is a solution to c_n.

The results of Blass, Hirst, and Simpson also hold for IHT.

One way to prove (I)HT is to use idempotent ultrafilters.

Iterated Hindman's Theorem (IHT): For instances $c_0, c_1, \ldots$ of HT, there are $x_0 < x_1 < \cdots$ s.t. each $\{x_i : i \geqslant n\}$ is a solution to c_n.

The results of Blass, Hirst, and Simpson also hold for IHT.

One way to prove (I)HT is to use idempotent ultrafilters.

Let $A - k = \{n : n + k \in A\}$.

The set of ultrafilters on $\mathbb{N}$ is a semigroup under the operation

$$\mathcal{U} \oplus \mathcal{V} = \{A : \{k : A - k \in \mathcal{U}\} \in \mathcal{V}\}.$$

$\mathcal{U}$ is **idempotent** if $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$.

Iterated Hindman's Theorem (IHT): For instances $c_0, c_1, \ldots$ of HT, there are $x_0 < x_1 < \cdots$ s.t. each $\{x_i : i \geqslant n\}$ is a solution to c_n.

The results of Blass, Hirst, and Simpson also hold for IHT.

One way to prove (I)HT is to use idempotent ultrafilters.

Let $A - k = \{n : n + k \in A\}$.

The set of ultrafilters on $\mathbb{N}$ is a semigroup under the operation

$$\mathcal{U} \oplus \mathcal{V} = \{A : \{k : A - k \in \mathcal{U}\} \in \mathcal{V}\}.$$

$\mathcal{U}$ is **idempotent** if $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$.

Hirst showed that IHT is equivalent to the existence of certain countable approximations to idempotent ultrafilters.

Montalbán and Shore developed a framework to expand the language of second-order arithmetic to talk about ultrafilters.

They showed that ACA_0 plus the existence of an idempotent ultrafilter implies IHT.

Montalbán and Shore developed a framework to expand the language of second-order arithmetic to talk about ultrafilters.

They showed that ACA_0 plus the existence of an idempotent ultrafilter implies IHT.

They also showed that the existence of an idempotent ultrafilter is conservative over $\mathsf{ACA}_0 + \mathsf{IHT}$, ACA_0^+, and several other systems.

Montalbán and Shore developed a framework to expand the language of second-order arithmetic to talk about ultrafilters.

They showed that ACA_0 plus the existence of an idempotent ultrafilter implies IHT.

They also showed that the existence of an idempotent ultrafilter is conservative over $\mathsf{ACA}_0 + \mathsf{IHT}$, ACA_0^+, and several other systems.

Kreuzer also showed the Π^1_2-conservativity of the existence of an idempotent ultrafilter over $\mathsf{ACA}_0 + \mathsf{IHT}$ and ACA_0^+ by working in higher-order reverse mathematics.

Montalbán and Shore developed a framework to expand the language of second-order arithmetic to talk about ultrafilters.

They showed that ACA_0 plus the existence of an idempotent ultrafilter implies IHT.

They also showed that the existence of an idempotent ultrafilter is conservative over $\mathsf{ACA}_0 + \mathsf{IHT}$, ACA_0^+, and several other systems.

Kreuzer also showed the Π^1_2-conservativity of the existence of an idempotent ultrafilter over $\mathsf{ACA}_0 + \mathsf{IHT}$ and ACA_0^+ by working in higher-order reverse mathematics.

Open Question. Is the existence of an idempotent ultrafilter conservative over ACA_0?

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

$\mathbf{HT}^{\leqslant \mathbf{n}}$ is HT for sums of at most n many elements.

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

$\mathbf{HT}^{\leqslant \mathbf{n}}$ is HT for sums of at most n many elements.

$\mathrm{HT}^{\leqslant 2}$ "should be" simpler to prove than HT, but:

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

HT$^{\leqslant n}$ is HT for sums of at most n many elements.

HT$^{\leqslant 2}$ "should be" simpler to prove than HT, but:

Open Question (Hindman, Leader, and Strauss). Is there a proof of HT$^{\leqslant 2}$ that is not a proof of HT?

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

$\mathbf{HT}^{\leqslant \mathbf{n}}$ is HT for sums of at most n many elements.

$\mathrm{HT}^{\leqslant 2}$ "should be" simpler to prove than HT, but:

Open Question (Hindman, Leader, and Strauss). Is there a proof of $\mathrm{HT}^{\leqslant 2}$ that is not a proof of HT? Does $\mathrm{HT}^{\leqslant 2} \to \mathrm{HT}$?

Hindman's Theorem (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite $S \subseteq \mathbb{N}$ s.t. every nonempty sum of distinct elements of S has the same color.

$\mathbf{HT}^{\leqslant \mathbf{n}}$ is HT for sums of at most n many elements.

$\mathrm{HT}^{\leqslant 2}$ "should be" simpler to prove than HT, but:

Open Question (Hindman, Leader, and Strauss). Is there a proof of $\mathrm{HT}^{\leqslant 2}$ that is not a proof of HT? Does $\mathrm{HT}^{\leqslant 2} \to \mathrm{HT}$?

Thm (Carlucci, Kołodziecczyk, Lepore, and Zdanowski).
$\mathrm{HT}^{\leqslant 2} \to \mathrm{ACA}_0$.

$\mathbf{HT}^{=\mathbf{n}}$ is HT for sums of exactly n many elements.

HT$^{=n}$ is HT for sums of exactly n many elements.

$[X]^n$ is the set of n-element subsets of X.

RT^n: For every coloring of $[\mathbb{N}]^n$ with finitely many colors, there is an infinite $H \subseteq \mathbb{N}$ s.t. every element of $[H]^n$ has the same color.

It is easy to see that $\mathrm{RT}^n \to \mathrm{HT}^{=n}$.

$\mathbf{HT^{=n}}$ is HT for sums of exactly n many elements.

$[X]^n$ is the set of n-element subsets of X.

$\mathbf{RT^n}$: For every coloring of $[\mathbb{N}]^n$ with finitely many colors, there is an infinite $H \subseteq \mathbb{N}$ s.t. every element of $[H]^n$ has the same color.

It is easy to see that $\mathrm{RT}^n \to \mathrm{HT}^{=n}$.

Thm (Seetapun). $\mathrm{RT}^2 \nrightarrow \mathrm{ACA}_0$.

So $\mathrm{HT}^{=2}$ is strictly weaker than ACA_0.

$\mathbf{HT^{=n}}$ is HT for sums of exactly n many elements.

$[X]^n$ is the set of n-element subsets of X.

$\mathbf{RT^n}$: For every coloring of $[\mathbb{N}]^n$ with finitely many colors, there is an infinite $H \subseteq \mathbb{N}$ s.t. every element of $[H]^n$ has the same color.

It is easy to see that $\mathrm{RT}^n \to \mathrm{HT}^{=n}$.

Thm (Seetapun). $\mathrm{RT}^2 \nrightarrow \mathrm{ACA}_0$.

So $\mathrm{HT}^{=2}$ is strictly weaker than ACA_0.

Question (Dzhafarov, Jockusch, Solomon, and Westrick). Is $\mathrm{HT}^{=2}$ computably true? Is it provable in RCA_0?

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). $HT^{=2}$ is not computably true, and hence is not provable in RCA_0.

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). $\mathrm{HT}^{=2}$ is not computably true, and hence is not provable in RCA_0.

Building a computable instance $c : \mathbb{N} \to 2$ of $\mathrm{HT}^{=2}$ with no computable solution:

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). $\mathsf{HT}^{=2}$ is not computably true, and hence is not provable in RCA_0.

Building a computable instance $c : \mathbb{N} \to 2$ of $\mathsf{HT}^{=2}$ with no computable solution:

Let $X + s = \{k + s : k \in X\}$ and let W_i be the i^{th} c.e. set.

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). $\mathsf{HT}^{=2}$ is not computably true, and hence is not provable in RCA_0.

Building a computable instance $c : \mathbb{N} \to 2$ of $\mathsf{HT}^{=2}$ with no computable solution:

Let $X + s = \{k + s : k \in X\}$ and let W_i be the i^{th} c.e. set.

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Ensure that $F_i + s$ is not monochromatic for all sufficiently large s.

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). $\mathsf{HT}^{=2}$ is not computably true, and hence is not provable in RCA_0.

Building a computable instance $c : \mathbb{N} \to 2$ of $\mathsf{HT}^{=2}$ with no computable solution:

Let $X + s = \{k + s : k \in X\}$ and let W_i be the i^{th} c.e. set.

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Ensure that $F_i + s$ is not monochromatic for all sufficiently large s.

Problem: interactions between the strategies for different i's.

Think of the $c(n)$'s as mutually independent random variables, with values 0 and 1 each having probability $\frac{1}{2}$.

Think of the $c(n)$'s as mutually independent random variables, with values 0 and 1 each having probability $\frac{1}{2}$.

If F_i is large then the event that $F_i + s$ is monochromatic for c has low probability.

Think of the $c(n)$'s as mutually independent random variables, with values 0 and 1 each having probability $\frac{1}{2}$.

If F_i is large then the event that $F_i + s$ is monochromatic for c has low probability.

These events for $F_i + s$ and $F_j + t$ are independent when s and t are sufficiently far apart.

Think of the $c(n)$'s as mutually independent random variables, with values 0 and 1 each having probability $\frac{1}{2}$.

If F_i is large then the event that $F_i + s$ is monochromatic for c has low probability.

These events for $F_i + s$ and $F_j + t$ are independent when s and t are sufficiently far apart.

So we need to know that when events with "sufficiently small" probability are "sufficiently independent" then it is possible to avoid them all

Think of the $c(n)$'s as mutually independent random variables, with values 0 and 1 each having probability $\frac{1}{2}$.

If F_i is large then the event that $F_i + s$ is monochromatic for c has low probability.

These events for $F_i + s$ and $F_j + t$ are independent when s and t are sufficiently far apart.

So we need to know that when events with "sufficiently small" probability are "sufficiently independent" then it is possible to avoid them all *effectively*.

To do this, we use the **Computable Lovász Local Lemma** of **Rumyantsev and Shen**, in the form of the following corollary:

To do this, we use the **Computable Lovász Local Lemma** of **Rumyantsev and Shen**, in the form of the following corollary:

For each $q \in (0, 1)$ there is an M s.t. the following holds.

Let $E_0, E_1, \ldots$ be a computable sequence of finite sets, each of size at least M.

Suppose that for each $m \geqslant M$ and n, there are at most 2^{qm} many i s.t. $|E_i| = m$ and $n \in E_i$, and that we can compute the set of all such i given m and n.

Then there is a computable $c : \mathbb{N} \to 2$ s.t. for each i the set E_i is not monochromatic for c.

Building a computable instance $c : \mathbb{N} \to 2$ of $\mathrm{HT}^{=2}$ with no computable solution:

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Use the computable LLL to ensure that $F_i + s$ is not monochromatic for all sufficiently large s.

Building a computable instance $c : \mathbb{N} \to 2$ of $\mathsf{HT}^{=2}$ with no computable solution:

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Use the computable LLL to ensure that $F_i + s$ is not monochromatic for all sufficiently large s.

We can work with $W_i^{\emptyset'}$ instead, to obtain a c with to Σ_2^0 solution.

Building a computable instance $c : \mathbb{N} \to 2$ of $\mathsf{HT}^{=2}$ with no computable solution:

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Use the computable LLL to ensure that $F_i + s$ is not monochromatic for all sufficiently large s.

We can work with $W_i^{\emptyset'}$ instead, to obtain a c with to Σ^0_2 solution.

The sizes of the F_i can be computably bounded, so we can also ensure that solutions to c are effectively immune relative to $\emptyset'$.

Building a computable instance $c : \mathbb{N} \to 2$ of $\mathrm{HT}^{=2}$ with no computable solution:

Wait for a sufficiently large finite $F_i \subseteq W_i$.

Use the computable LLL to ensure that $F_i + s$ is not monochromatic for all sufficiently large s.

We can work with $W_i^{\emptyset'}$ instead, to obtain a c with to Σ^0_2 solution.

The sizes of the F_i can be computably bounded, so we can also ensure that solutions to c are effectively immune relative to $\emptyset'$.

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). There is a computable instance of $\mathrm{HT}^{=2}$ s.t. every solution is diagonally noncomputable (DNC) relative to $\emptyset'$.

2-DNC is the reverse-mathematical principle corresponding to diagonal noncomputability relative to the jump.

2-DNC is the reverse-mathematical principle corresponding to diagonal noncomputability relative to the jump.

$\mathbf{RRT}^2_2$: If $c : [\mathbb{N}]^2 \to \mathbb{N}$ is s.t. $|c^{-1}(i)| \leqslant 2$ for all i, there there is an infinite $R \subseteq \mathbb{N}$ s.t. c is injective on $[R]^2$.

Thm (J. Miller). $\mathrm{RRT}^2_2 \leftrightarrow$ 2-DNC.

2-DNC is the reverse-mathematical principle corresponding to diagonal noncomputability relative to the jump.

RRT^2_2: If $c : [\mathbb{N}]^2 \to \mathbb{N}$ is s.t. $|c^{-1}(i)| \leqslant 2$ for all i, there there is an infinite $R \subseteq \mathbb{N}$ s.t. c is injective on $[R]^2$.

Thm (J. Miller). $\mathsf{RRT}^2_2 \leftrightarrow$ 2-DNC.

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). $\mathsf{HT}^{=2} \to$ 2-DNC.

2-DNC is the reverse-mathematical principle corresponding to diagonal noncomputability relative to the jump.

RRT^2_2: If $c : [\mathbb{N}]^2 \to \mathbb{N}$ is s.t. $|c^{-1}(i)| \leqslant 2$ for all i, there there is an infinite $R \subseteq \mathbb{N}$ s.t. c is injective on $[R]^2$.

Thm (J. Miller). $\mathsf{RRT}^2_2 \leftrightarrow$ 2-DNC.

Thm (Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick). $\mathsf{HT}^{=2} \to$ 2-DNC.

Open Question. Does 2-DNC $\to \mathsf{HT}^{=2}$?

Open Question. Does $\mathsf{HT}^{=2} \to \mathsf{RT}^2$?

An instance of $\mathrm{HT}^{\leqslant 2}$ or RT^2 might have no solution containing a given n.

An instance of $\mathsf{HT}^{\leqslant 2}$ or RT^2 might have no solution containing a given n.

However, every instance of $\mathsf{HT}^{=2}$ does have such a solution:

An instance of $\mathsf{HT}^{\leqslant 2}$ or RT^2 might have no solution containing a given n.

However, every instance of $\mathsf{HT}^{=2}$ does have such a solution:

$\mathbf{HT^{=2}(n)}$: every instance of $\mathsf{HT}^{=2}$ has a solution containing n.

An instance of $\mathsf{HT}^{\leqslant 2}$ or RT^2 might have no solution containing a given n.

However, every instance of $\mathsf{HT}^{=2}$ does have such a solution:

$\mathbf{HT}^{=2}(\mathbf{n})$: every instance of $\mathsf{HT}^{=2}$ has a solution containing n.

$\mathsf{HT}^{=2}(0)$ is basically $\mathsf{HT}^{\leqslant 2}$.

An instance of $\mathsf{HT}^{\leqslant 2}$ or RT^2 might have no solution containing a given n.

However, every instance of $\mathsf{HT}^{=2}$ does have such a solution:

$\mathbf{HT}^{=\mathbf{2}}(\mathbf{n})$: every instance of $\mathsf{HT}^{=2}$ has a solution containing n.

$\mathsf{HT}^{=2}(0)$ is basically $\mathsf{HT}^{\leqslant 2}$.

We can pass between $\mathsf{HT}^{=2}(0)$ and $\mathsf{HT}^{=2}(n)$ by translating the coloring by $2n$ and then translating the solution back by n.

Thus every $\mathsf{HT}^{=2}(n)$ is equivalent to $\mathsf{HT}^{\leqslant 2}$.

HT is equivalent to the **Finite Union Theorem (FUT)**: For every coloring of the finite subsets of $\mathbb{N}$ with finitely many colors, there are nonempty finite sets $F_0 < F_1 < F_2 < \cdots$ such that all nonempty finite unions of the F_i's have the same color.

HT is equivalent to the **Finite Union Theorem (FUT)**: For every coloring of the finite subsets of $\mathbb{N}$ with finitely many colors, there are nonempty finite sets $F_0 < F_1 < F_2 < \cdots$ such that all nonempty finite unions of the F_i's have the same color.

Hirst considered the following variation, motivated by a lemma of Hilbert:

HIL: For every coloring of the finite subsets of $\mathbb{N}$ with finitely many colors, there are distinct nonempty finite sets $F_0, F_1, F_2, \ldots$ such that all nonempty finite unions of the F_i's have the same color.

HT is equivalent to the **Finite Union Theorem (FUT)**: For every coloring of the finite subsets of $\mathbb{N}$ with finitely many colors, there are nonempty finite sets $F_0 < F_1 < F_2 < \cdots$ such that all nonempty finite unions of the F_i's have the same color.

Hirst considered the following variation, motivated by a lemma of Hilbert:

HIL: For every coloring of the finite subsets of $\mathbb{N}$ with finitely many colors, there are distinct nonempty finite sets $F_0, F_1, F_2, \ldots$ such that all nonempty finite unions of the F_i's have the same color.

Thm (Hirst). HIL $\leftrightarrow$ RT^1.

Thus HIL is computably true (though not quite provable in RCA_0).

Let P be a version of HT.

$\mathbf{P_k}$ is P restricted to k-colorings.

Let P be a version of HT.

$\mathbf{P_k}$ is P restricted to k-colorings.

Let $\lambda(n)$ be the least exponent of n base 2, and let $\mu(n)$ be the greatest exponent of n base 2.

$S \subseteq \mathbb{N}$ **satisfies apartness** if for all $m < n$ in S, we have $\mu(m) < \lambda(n)$.

P with apartness is P with the extra condition that the solution satisfy apartness.

Let P be a version of HT.

P_k is P restricted to k-colorings.

Let $\lambda(n)$ be the least exponent of n base 2, and let $\mu(n)$ be the greatest exponent of n base 2.

$S \subseteq \mathbb{N}$ **satisfies apartness** if for all $m < n$ in S, we have $\mu(m) < \lambda(n)$.

P with apartness is P with the extra condition that the solution satisfy apartness.

Thinking of HT as FUT makes apartness natural.

HT_k and HT_k with apartness are equivalent to FUT_k and hence to each other.

Thm (Carlucci, Kołodzieczyk, Lepore, and Zdanowski).

1. $\mathsf{HT}_k^{\leqslant n}$ with apartness is equivalent to $\mathsf{FUT}_k^{\leqslant n}$, and also for $=n$.

Thm (Carlucci, Kołodzieczyk, Lepore, and Zdanowski).

1. $\mathrm{HT}_k^{\leqslant n}$ with apartness is equivalent to $\mathrm{FUT}_k^{\leqslant n}$, and also for $=n$.
2. $\mathrm{HT}_{2k}^{\leqslant n}$ implies $\mathrm{HT}_k^{\leqslant n}$ with apartness.

Thm (Carlucci, Kołodzieczyk, Lepore, and Zdanowski).

1. $\mathsf{HT}_k^{\leqslant n}$ with apartness is equivalent to $\mathsf{FUT}_k^{\leqslant n}$, and also for $=n$.

2. $\mathsf{HT}_{2k}^{\leqslant n}$ implies $\mathsf{HT}_k^{\leqslant n}$ with apartness.

3. $\mathsf{HT}_2^{\leqslant 2}$ with apartness implies ACA_0.

4. $\mathsf{HT}_4^{\leqslant 2}$ implies ACA_0.

Thm (Carlucci, Kołodziecczyk, Lepore, and Zdanowski).

1. $\mathrm{HT}_k^{\leqslant n}$ with apartness is equivalent to $\mathrm{FUT}_k^{\leqslant n}$, and also for $=n$.
2. $\mathrm{HT}_{2k}^{\leqslant n}$ implies $\mathrm{HT}_k^{\leqslant n}$ with apartness.
3. $\mathrm{HT}_2^{\leqslant 2}$ with apartness implies ACA_0.
4. $\mathrm{HT}_4^{\leqslant 2}$ implies ACA_0.
5. For $n \geqslant 3$, $\mathrm{HT}_k^{=n}$ with apartness is equivalent to ACA_0.

Thm (Carlucci, Kołodzieczyk, Lepore, and Zdanowski).

1. $\mathsf{HT}_k^{\leqslant n}$ with apartness is equivalent to $\mathsf{FUT}_k^{\leqslant n}$, and also for $=n$.
2. $\mathsf{HT}_{2k}^{\leqslant n}$ implies $\mathsf{HT}_k^{\leqslant n}$ with apartness.
3. $\mathsf{HT}_2^{\leqslant 2}$ with apartness implies ACA_0.
4. $\mathsf{HT}_4^{\leqslant 2}$ implies ACA_0.
5. For $n \geqslant 3$, $\mathsf{HT}_k^{=n}$ with apartness is equivalent to ACA_0.

Thm (Dzhafarov, Jockusch, Solomon, and Westrick).

1. $\mathsf{HT}_3^{\leqslant 3}$ implies ACA_0.
2. $\mathsf{HT}_2^{\leqslant 2}$ implies the stable version of RT_2^2 over $\mathsf{B}\Sigma_2^0$.