SOME RECENT NEWS ABOUT TRUTH THEORIES

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Base theories and Truth theories

- A base theory B is a first order theory with "enough coding for handling finite sequences of objects", for example:
  (1) B = PA (Peano arithmetic).
  (2) B = ACA_0 (the predicative extension of PA).
  (3) B = ZF (Zermelo-Fraenkel set theory).
  (4) B = GB (the predicative extension of ZF).
- A truth theory over a base theory B is a theory of the form:

$$\mathrm{P[B]} = \mathrm{B} \cup \mathrm{P},$$

  where P (for prawda = truth in Polish) is a set of "truth axioms" formulated in the language $\mathcal{L}_{\mathrm{B}} \cup \{\mathrm{T}(x)\}$, where the intended interpretation of $\mathrm{T}(x)$ is "x is the Gödel number of a true $\mathcal{L}_{\mathrm{B}}$-sentence".
- We will refer to $\mathcal{L}_{\mathrm{B}} \cup \{\mathrm{T}\}$ as "the extended language".

Compositional Truth

- CT^- consists of the following axioms, where s and t range over closed terms of $\mathcal{L}_{\mathrm{B}}$; and ϕ and ψ range over formulae of $\mathcal{L}_{\mathrm{B}}$.

CT1 $T(s=t) \leftrightarrow \left(s^\circ = t^\circ\right)$.

CT2 $\left\{T\left(R(s_1,\ldots,s_n)\right) \leftrightarrow R({s_1}^\circ,\ldots,{s_n}^\circ) : R \in \mathcal{L}_{\mathrm{B}}\right\}$.

CT3 $T(\neg\phi) \leftrightarrow \neg T(\phi)$.

CT4 $T(\phi \vee \psi) \leftrightarrow T(\phi) \vee T(\psi)$.

CT5 $T(\exists v\ \phi) \leftrightarrow \exists x T(\phi(\underline{x}))$.

CT6 $\left(\bar{s}^\circ = \bar{t}^\circ \rightarrow T(\phi(\bar{s})) \leftrightarrow T(\phi(\bar{t}))\right)$.

- $\mathrm{CT}^- \vdash \mathrm{TB}^-$, where TB^- consist of Tarski bi-conditionals

$$\mathrm{T}(\phi) \leftrightarrow \phi$$

Friedman-Sheard untyped truth theory

FS^- consists of the following axioms, where s and t range over closed terms of $\mathcal{L}_{\mathrm{B}}$; and ϕ and ψ range over formulae of $\mathcal{L}_{\mathrm{B}} \cup \{\mathrm{T}(x)\}$.

FS1 $T(s = t) \leftrightarrow \big(s^\circ = t^\circ\big)$.

FS2 $\{T\big(R(s_1, \ldots, s_n)\big) \leftrightarrow R({s_1}^\circ, \ldots, {s_n}^\circ) : R \in \mathcal{L}_{\mathrm{B}}\}$.

FS3 $T(\neg\phi) \leftrightarrow \neg T(\phi)$.

FS4 $T(\phi \vee \psi) \leftrightarrow T(\phi) \vee T(\psi)$.

FS5 $T(\exists v\ \phi) \leftrightarrow \exists x T(\phi(\underline{x}))$.

FS6 $\Big(\bar{s}^\circ = \bar{t}^\circ \to T(\phi(\bar{s})) \leftrightarrow T(\phi(\bar{t}))\Big)$.

FS^- is equipped with the following additional derivation rules:

$$\frac{\phi}{T(\phi)}\ \text{(NEC)} \qquad\qquad \frac{T(\phi)}{\phi}\ \text{(CONEC)}$$

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Kripke-Feferman untyped truth theory

- KF^- consists of the following axioms, where s and t range over closed terms of $\mathcal{L}_{\mathrm{B}}$; and ϕ and ψ range over formulae of $\mathcal{L}_{\mathrm{B}} \cup \{\mathrm{T}(x)\}$.

KF1 $T(s = t) \leftrightarrow (s^\circ = t^\circ)$.

KF2 $T(s \neq t) \leftrightarrow (s^\circ \neq t^\circ)$.

KF3 $\{T\big(R(s_1, \ldots, s_n)\big) \leftrightarrow R(s_1{}^\circ, \ldots, s_n{}^\circ) : R \in \mathcal{L}_{\mathrm{B}}\}$.

KF4 $\{T\big(\neg R(s_1, \ldots, s_n)\big) \leftrightarrow \neg R(s_1{}^\circ, \ldots, s_n{}^\circ) : R \in \mathcal{L}_{\mathrm{B}}\}$.

KF5 $T(\neg\neg\phi) \leftrightarrow T(\phi)$.

KF6 $T(\phi \vee \psi) \leftrightarrow T(\phi) \vee T(\psi)$.

KF7 $T(\neg(\phi \vee \psi)) \leftrightarrow T(\neg\phi) \wedge T(\neg\psi)$.

KF8 $T(\exists y\ \phi(y)) \leftrightarrow \exists x T(\phi(\underline{x}))$.

KF9 $T(\neg\exists y\ \phi(y)) \leftrightarrow \forall x T(\neg\phi(\underline{x}))$.

KF10 $\Big(\bar{s}^\circ = \bar{t}^\circ \rightarrow T(\phi(\bar{s})) \leftrightarrow T(\phi(\bar{t}))\Big)$.

KF11 $\Big(t^\circ = \phi \rightarrow T(T(t)) \leftrightarrow T(\phi)\Big)$.

KF12 $\Big(t^\circ = \phi \rightarrow T(\neg T(t)) \leftrightarrow T(\neg\phi)\Big)$.

The power of truth axioms in the presence of induction

- Given a truth theory P^-, let P be the result of strengthening P^- with all instances of induction in the extended language.
- (Tarski, 1935) CT[PA] proves Con(PA), but TB[PA] is conservative over PA.
- (Feferman, 1964) The arithmetical consequences of CT[PA] coincide with the arithmetical consequences of ACA.
- (Halbach, 2010) The arithmetical consequences of FS coincide with the arithmetical consequences of $\mathsf{RA}_{<\omega}$.
- (Feferman 1985, Cantini 1987) The arithmetical consequences of KF coincide with the arithmetical consequences of $\mathsf{RA}_{<\varepsilon_0}$.

A little induction goes very far

- Let CT_n be the fragment of CT[PA] in which the scheme of induction for formulae in the extended language is restricted to Σ_n formulae.
- $\mathrm{CT}_1 \vdash \mathrm{GR}_{\mathrm{PA}}$, where $\mathrm{GR}_{\mathrm{PA}}$ is the sentence in the extended language expressing: "all theorems of PA are true". Therefore $\mathrm{CT}_1 \vdash \mathrm{Con(PA)}$, since $\mathrm{CT}^- \vdash \neg \mathrm{T}(0 = 1)$.
- What about CT_0? In a 1986 paper, Kotlarski claimed that CT_0 proves $\mathrm{GR}_{\mathrm{PA}}$, but a serious gap was found in his proof outline around 2012 by Heck and Visser.
- Łełyk and Wcisło (2017) proved that CT_0 and $\mathrm{GR}_{\mathrm{PA}}$ have the same arithmetic consequences.
- Łełyk (PhD thesis, 2018) confirmed Kotlarski's hunch by verifying that $\mathrm{GR}_{\mathrm{PA}}$ is provable in CT_0.

Conservativity of typed truth

- Krajewski-Kotlarski-Lachlan (1981) showed that $\mathsf{CT}^-[\mathsf{PA}] + \mathsf{IC}$ is conservative over PA, where IC is the axiom of Induction Correctness (also known as Internal Induction) asserting, as one sentence, that each instance of arithmetical induction is true.
- We now know that, more generally, $\mathsf{CT}^-[\mathsf{B}]$ is conservative over B for every base theory B, and that if τ is a "scheme template" such that B proves every instance of τ, then:
  $\mathsf{CT}^-[\mathsf{B}] + \text{"every instance of } \tau \text{ is true"}$ is conservative over B.
- These general results were established in the joint work of Visser and me (2014), using elementary model-theoretic ideas, and by Leigh (2015), using proof-theoretic machinery.

Conservativity of untyped truth

- Cantini (1989) showed that $KF^{-}[B]$ is conservative over B.

- Halbach (2011) noted that the conservativity of $FS^{-}[B]$ over B can be established by using:

  (1) the conservativity of $CT^{-}[B]$ over B for all B, and

  (2) the proof theoretic reduction of FS[PA] to $RT_{<\omega}$.

Between CT^- and CT_0

- Cieślinski (2010) and (2017) showed:
  (1) $CT^-PA] + VAL \subseteq T$ proves CT_0, where $VAL \subseteq T$ expresses:
  "T contains all arithmetical instances of theorems of first order logic".
  (2) $CT^-[PA] + Cl_{Prop(T)} \subseteq T$ proves CT_0, where $Cl_{Prop}(T) \subseteq T$ expresses:
  "T is closed under propositional logic".
- It is easy to verify that (1) $CT_0 \vdash VAL \subseteq T$,
  (2) $CT^- \vdash VAL \subseteq T \rightarrow Cl_{Prop}(T) \subseteq T$, and
  (3) $CT^-[PA] + Cl_{Prop} \subseteq T \vdash DC \wedge IC$, where
  DC (Disjunctive Correctness) is the axiom that expresses: a finite disjunction of arithmetical sentences is true iff one of the disjuncts is true.
- Open Question 1. Is $CT^- + VAL_{Prop} \subseteq T$ conservative over PA?
- I showed (2012) that $CT^-[PA] + DC + IC \vdash CT_0$.

The surprising power of Disjunctive Correctness (1)

- **Theorem** (Pakhomov, (2019)) $\mathsf{CT}^-[\mathsf{PA}] + \mathsf{DC} \vdash \mathsf{IC}$.
- Coupled with the facts that (1) $\mathsf{CT}^-[\mathsf{PA}] + \mathsf{DC} + \mathsf{IC}$ proves CT_0, and (2) CT_0 proves Con (PA), and (3) Gödel's second incompleteness theorem, Pakhomov's theorem shows that $\mathsf{CT}^-[\mathsf{PA}] + \mathsf{DC}$ is not conservative over PA.
- Pakhomov's proof is based on a generalization of Visser's theorem on the non-existence of infinite descending chains of truth definitions. Its proof uses (Löb's version) of Gödel's second incompleteness theorem, and therefore implicitly uses the Gödel-Carnap fixed point theorem.
- **Proof Outline.** Consider the theory ITB (iterated truth biconditionals) extending Robinson's Q + "$\prec$ is a transitive relation" plus the following biconditionals B_φ:

$$\mathsf{B}_\varphi := \forall\alpha(\mathsf{T}_\alpha(\ulcorner\phi\urcorner) \leftrightarrow \phi^{\prec\alpha}).$$

Lemma (1) *The following theory* DTB *is inconsistent:*

$$\mathsf{DTB} := \mathsf{ITB} + \forall\alpha\exists\beta(\beta \prec \alpha) + \exists\alpha(\alpha = \alpha).$$

The surprising power of Disjunctive Correctness (2)

Lemma (2) *Every finite subtheory of* DTB *is interpretable in* $\mathrm{CT}^-[\mathrm{I}\Delta_0 + \mathrm{Exp}] + \mathrm{DC} + \neg\, \mathrm{IC}$.

- Pakhomov later found a proof based on a result of Flumini and Sato concerning the relationship between well-foundedness and the existence of hierarchies in second order systems (2014) whose proof is fixed-point free.
- DC can be written as $\mathrm{DC}_{\mathrm{Elim}} + \mathrm{DC}_{\mathrm{Intro}}$.
- Recent joint work of Wcisło, Łełyk and E. shows that $\mathrm{DC}_{\mathrm{Intro}}$ can be conservatively added to: $\mathrm{CT}^-[\mathrm{PA}] + \mathrm{IC} + \{\forall x\, (\mathrm{True}_{\Sigma_n}(x) \rightarrow \mathrm{T}(x)) : n < \omega\}$.
- Open Problem 2. Is $\mathrm{CT}^-[\mathrm{PA}] + \mathrm{DC}_{\mathrm{Elim}}$ *conservative over* PA?
- Open Problem 3. Is $\mathrm{CT}^-[\mathrm{S}^1_2] + \mathrm{DC}$ *conservative over* PA?

How complex is the reduction of P[B] to B?

- Suppose P[B] is conservative over B. Among the commonly studied computational classes of functions $\mathcal{F}$, what is the *optimal complexity class* $\mathcal{F}$ that contains some f with the property that for all proofs π and all $\mathcal{L}_{\mathsf{B}}$-sentences ϕ, we have:

$$\mathsf{P[B]} \vdash_{\pi} \phi \Longrightarrow \mathsf{B} \vdash_{f(\pi)} \phi.$$

- Let Supexp-time be the class of functions that are computable by a Turing machine whose run time is bounded above by a function that is provably total in the fragment of PRA known as SEFA. An examination of Leigh's 2015 proof makes it clear that there is a Supexp-time computable function f such that for all proofs π and all $\mathcal{L}_{\mathsf{B}}$-sentences ϕ we have:

$$\mathsf{CT}^{-}[\mathsf{B}] \vdash_{\pi} \phi \Longrightarrow \mathsf{B} \vdash_{f(\pi)} \phi.$$

Examples where the Supexp upper bound is optimal

- By a theorem of Pudlák (1985), if T_1 is a sequential theory, and $T_2 \supseteq T_1$ and T_2 proves the consistency of T_1 on a cut, then T_2 has superexponential speed-up over T_1.
- Pudlák's above theorem readily implies that $\mathrm{CT}^-[\mathrm{B}]$ is not Exp-time reducible to B if B is finitely axiomatizable. Therefore, for finitely axiomatizable B, Leigh's Supexp upper bound is optimal.
- Pudlák's theorem also implies that Supexp is optimal for reducing $\mathrm{CT}^-[\mathrm{PA}] + \mathrm{IC}$ to PA.
- What about $\mathrm{CT}^-[\mathrm{PA}]$?

Feasible interpretations

- $I : \mathsf{B} \rhd \mathsf{P[B]}$ is a feasible interpretation if there is a P-time computable function $f(s)$ such that for all proofs π and all $\mathcal{L}_{\mathsf{P[B]}}$-formulae ϕ,

$$\mathsf{P[B]} \vdash_{\pi} \phi \Longrightarrow \mathsf{B} \vdash_{f(\pi)} \phi^{I}.$$

- Feasible interpretations were first systematically studied in the 1993 doctoral dissertation of Rineke Verbrugge. She showed, among many other things, that:

There is a sentence θ such that $\mathsf{PA} \rhd \mathsf{PA} + \theta$, but $\mathsf{PA} \not\rhd_{f} \mathsf{PA} + \theta$.

Feasibly neat interpretations (1)

- A family of interpretations $\{I_n\}_{n\in\mathbb{N}} : \mathrm{B} \rhd \mathrm{P[B]}$ is feasibly neat if there are P-time computable functions $f(s_0, s_1)$ and $g(s_0, s_1)$ such that the following two conditions hold:

  1. For every $k \in \mathbb{N}$, and every $\mathcal{L}_\mathrm{B}$-formula ϕ of length at most k,

  $$\mathrm{B} \vdash_{f(\mathrm{tal}(k),\phi)} \phi^{I_k} \rightarrow \phi,$$

  where $\mathrm{tal}(k)$ is the tally numeral $1+1+...+1$ (k times).

  2. For every $k \in \mathbb{N}$, and every proof π,

  $$\mathrm{P[B]} \vdash_\pi \phi \Longrightarrow \mathrm{B} \vdash_{g(\mathrm{tal}(k),\pi)} \phi^{I_k}.$$

Feasibly neat interpretations (2)

- **Theorem** (Joint with Mateusz Łełyk and Bartosz Wcisło) (2019). *Let* P *denote any of the truth theories* CT^{-}, FS^{-}, *and* KF^{-}. *There is a family* $\{I_n\}_{n\in\mathbb{N}} : \mathsf{PA} \rhd \mathsf{P}[\mathsf{PA}]$ *of feasibly neat interpretations.*

Corollaries of the Theorem

- Let P denote any of the truth theories CT^-, FS^- *and* KF^-.
- **Corollary 1.** P[PA] *is feasibly reducible to* PA *,i.e., there is a polynomial-time computable function* f *with the property that for all proofs* π *and all* $\mathcal{L}_{\mathrm{PA}}$*-sentences* ϕ*, we have*:

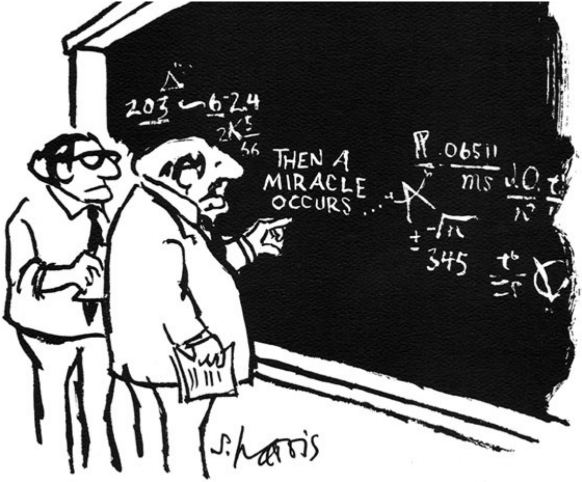
$$\mathrm{P[PA]} \vdash_{\pi} \phi \Longrightarrow \mathrm{PA} \vdash_{f(\pi)} \phi.$$

- **Corollary 2.** P[PA] *has at most polynomial speed-up over* PA, *i.e., there is a polynomial* $p(n)$ *such that for all* $n \in \mathbb{N}$

$$\mathrm{P[PA]} \vdash^{\leq n} \phi \Longrightarrow \mathrm{PA} \vdash^{\leq p(n)} \phi.$$

- Open Problem 4. Is the conservativity of $\mathrm{CT}^-[\mathrm{PA}]$ *over* PA *verifiable in* S^1_2?

Thank you for your attention



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."