Local proof-theoretic foundations and proof-theoretic tameness in ordinary mathematics

Ulrich Kohlenbach

Department of Mathematics

TECHNISCHE UNIVERSITÄT DARMSTADT

ASL Logic Colloquium 2019, Prague, August 15, 2019

Retiring Presidential Address

Unwinding of proofs and Proof Mining

- Starting in the 50's Kreisel proposed to use proof-theoretic transformations (developed in pursuing Hilbert's program) to extract ('unwind') new information (bounds etc.) from interesting given proofs.

Unwinding of proofs and Proof Mining

- Starting in the 50's Kreisel proposed to use proof-theoretic transformations (developed in pursuing Hilbert's program) to extract ('unwind') new information (bounds etc.) from interesting given proofs.
- Early 'unwindings' concerned mainly algebra, number theory, combinatorics (Kreisel, Delzell, Girard, Macintyre, Luckhardt).

Unwinding of proofs and Proof Mining

- Starting in the 50's Kreisel proposed to use proof-theoretic transformations (developed in pursuing Hilbert's program) to extract ('unwind') new information (bounds etc.) from interesting given proofs.
- Early 'unwindings' concerned mainly algebra, number theory, combinatorics (Kreisel, Delzell, Girard, Macintyre, Luckhardt).
- Since 90's mainly applications in analysis ('proof mining')

Proof Mining since 2000 (abstract classes of spaces)

Since K. NFAO 2001: applications of proof mining in analysis with **abstract metric spaces** involved: **fixed point and ergodic theory, convex optimization, geodesic geometry, Cauchy problems, game theory etc.**

Proof Mining since 2000 (abstract classes of spaces)

Since K. NFAO 2001: applications of proof mining in analysis with **abstract metric spaces** involved: **fixed point and ergodic theory, convex optimization, geodesic geometry, Cauchy problems, game theory etc.**

Setting: X normed linear or geodesic space, $\boldsymbol{C} \subseteq \boldsymbol{X}$ some (often convex) subset and $\boldsymbol{T : C \to C}$ or $\boldsymbol{F : C} \to \overline{\mathbb{R}}$ nonlinear maps.

Proof Mining since 2000 (abstract classes of spaces)

Since K. NFAO 2001: applications of proof mining in analysis with **abstract metric spaces** involved: **fixed point and ergodic theory, convex optimization, geodesic geometry, Cauchy problems, game theory etc.**

Setting: X normed linear or geodesic space, $C \subseteq X$ some (often convex) subset and $T : C \to C$ or $F : C \to \overline{\mathbb{R}}$ nonlinear maps.

Covers numerous fixed point, zero-finding, minimization or equilibirium problems with iterative procedures (x_n) s.t. e.g. in the case of fixed point problems one has

$$\textbf{(1)}\ d(x_n, Tx_n) \overset{n\to\infty}{\to} = 0 \quad \text{or even}$$

(2) (x_n) strongly converges to the fixed point of T.

For such situations, **special designed** (for **particular classes** of spaces X and mappings T) **logical metatheorems** (K. TAMS 2005, Gerhardy/K. TAMS 2008) have been designed which guarantee the extractability of explicit uniform bounds for $\forall \underline{x} \in \mathbb{N}, \mathbb{N}^{\mathbb{N}}, X, X^X, X^{\mathbb{N}} \ldots \exists n \in \mathbb{N}\, A(\underline{x}, n)$-theorems.

For such situations, **special designed** (for **particular classes** of spaces X and mappings T) **logical metatheorems** (K. TAMS 2005, Gerhardy/K. TAMS 2008) have been designed which guarantee the extractability of explicit uniform bounds for $\forall \underline{x} \in \mathbb{N}, \mathbb{N}^{\mathbb{N}}, X, X^X, X^{\mathbb{N}} \ldots \exists n \in \mathbb{N}\, A(\underline{x}, n)$-theorems.

Due to **classical logic**: in general A must be **existential** $A_{\exists}$.

For such situations, **special designed** (for **particular classes** of spaces X and mappings T) **logical metatheorems** (K. TAMS 2005, Gerhardy/K. TAMS 2008) have been designed which guarantee the extractability of explicit uniform bounds for $\forall \underline{x} \in \mathbb{N}, \mathbb{N}^{\mathbb{N}}, X, X^X, X^{\mathbb{N}} \ldots \exists n \in \mathbb{N}\, A(\underline{x}, n)$-theorems.

Due to **classical logic**: in general A must be **existential** $A_{\exists}$.

For **restricted use of LEM** **arbitrary** A**!**

For such situations, **special designed** (for **particular classes** of spaces X and mappings T) **logical metatheorems** (K. TAMS 2005, Gerhardy/K. TAMS 2008) have been designed which guarantee the extractability of explicit uniform bounds for $\forall \underline{x} \in \mathbb{N}, \mathbb{N}^{\mathbb{N}}, X, X^X, X^{\mathbb{N}} \ldots \exists n \in \mathbb{N}\, A(\underline{x}, n)$-theorems.

Due to **classical logic**: in general A must be **existential** $A_\exists$.

For **restricted use of LEM arbitrary** A**!**

The **logical metatheorems** guarantee the extractability of effective bounds on '∃' **independent** from parameters in

For such situations, **special designed** (for **particular classes** of spaces X and mappings T) **logical metatheorems** (K. TAMS 2005, Gerhardy/K. TAMS 2008) have been designed which guarantee the extractability of explicit uniform bounds for $\forall \underline{x} \in \mathbb{N}, \mathbb{N}^{\mathbb{N}}, X, X^X, X^{\mathbb{N}} \ldots \exists n \in \mathbb{N}\, A(\underline{x}, n)$-theorems.

Due to **classical logic**: in general A must be **existential** $A_{\exists}$.

For **restricted use of LEM arbitrary** A**!**

The **logical metatheorems** guarantee the extractability of effective bounds on '∃' **independent** from parameters in

- **compact** metric spaces (**if separability is used**) and

For such situations, **special designed** (for **particular classes** of spaces X and mappings T) **logical metatheorems** (K. TAMS 2005, Gerhardy/K. TAMS 2008) have been designed which guarantee the extractability of explicit uniform bounds for $\forall \underline{x} \in \mathbb{N}, \mathbb{N}^{\mathbb{N}}, X, X^X, X^{\mathbb{N}} \ldots \exists n \in \mathbb{N}\, A(\underline{x}, n)$-theorems.

Due to **classical logic**: in general A must be **existential** $A_\exists$.

For **restricted use of LEM arbitrary** A**!**

The **logical metatheorems** guarantee the extractability of effective bounds on '$\exists$' **independent** from parameters in

- **compact** metric spaces (**if separability is used**) and
- bounded subsets of **abstract** metric structures X.

Formal systems for analysis with abstract metric spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Formal systems for analysis with abstract metric spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

PA$^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**Dependent choice in all types**.

Formal systems for analysis with abstract metric spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

PA$^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**Dependent choice in all types**.

Implies **full comprehension** for numbers (2nd order arithmetic).

In practice: only **weak fragments** needed (after pre-processing).

Formal systems for analysis with abstract metric spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

PA$^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**Dependent choice in all types**.

Implies **full comprehension** for numbers (2nd order arithmetic).

In practice: only **weak fragments** needed (after pre-processing).

Equality is a **defined** notion: $x^X =_X y^X :\equiv d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}}$.

Formal systems for analysis with abstract metric spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

PA$^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**Dependent choice in all types**.

Implies **full comprehension** for numbers (2nd order arithmetic).

In practice: only **weak fragments** needed (after pre-processing).

Equality is a **defined** notion: $x^X =_X y^X :\equiv d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}}$.

In general only **rule**

$$\frac{s =_X t}{T(s) =_X T(t)}.$$

Formal systems for analysis with abstract metric spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

PA$^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**Dependent choice in all types**.

Implies **full comprehension** for numbers (2nd order arithmetic).

In practice: only **weak fragments** needed (after pre-processing).

Equality is a **defined** notion: $x^X =_X y^X :\equiv d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}}$.

In general only **rule**

$$\frac{s =_X t}{T(s) =_X T(t)}.$$

$\mathcal{A}^{\omega}[X, \|\cdot\| \ldots]$ e.g. results by adding constants with axioms expressing that $(X, \|\cdot\|)$ is normed, uniformly convex, Hilbert.

Special case of **general logical metatheorems** (T nonexpansive):

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, \|\cdot\|]$ proves ('n.e.' means 'nonexpansive')

$$\forall n \in \mathbb{N}\, \forall x \in X\, \forall T : X \to X\, \big(T \text{ n.e.} \to \exists k \in \mathbb{N}\, A_\exists\big),$$

Special case of **general logical metatheorems** (T nonexpansive):

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, \|\cdot\|]$ proves ('n.e.' means 'nonexpansive')

$$\forall n \in \mathbb{N}\, \forall x \in X\, \forall T : X \to X\ \big(T \text{ n.e.} \to \exists k \in \mathbb{N}\, A_\exists\big),$$

then one can extract a **computable function** $\Phi : \mathbb{N}^2 \to \mathbb{N}$ s.t. in all normed spaces X it holds that

$$\forall n, b \in \mathbb{N}\, \forall x \in X\, \forall T : X \to X$$
$$\big(T \text{ n.e.} \wedge \|x\|, \|T(0)\| \leq b \to \exists k \leq \Phi(n, b)\, A_\exists\big).$$

Special case of **general logical metatheorems** (T nonexpansive):

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, \|\cdot\|]$ proves ('n.e.' means 'nonexpansive')

$$\forall n \in \mathbb{N} \, \forall x \in X \, \forall T : X \to X \, \big(T \text{ n.e.} \to \exists k \in \mathbb{N} \, A_\exists\big),$$

then one can extract a **computable function** $\Phi : \mathbb{N}^2 \to \mathbb{N}$

s.t. in all normed spaces X it holds that

$$\forall n, b \in \mathbb{N} \, \forall x \in X \, \forall T : X \to X$$
$$\big(T \text{ n.e.} \wedge \|x\|, \|T(0)\| \leq b \to \exists k \leq \Phi(n, b) \, A_\exists\big).$$

Similar for Hilbert spaces and uniformly convex spaces (then bound depends on modulus of convexity).

Special case of **general logical metatheorems** (T nonexpansive):

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, \|\cdot\|]$ proves ('n.e.' means 'nonexpansive')

$$\forall n \in \mathbb{N}\, \forall x \in X\, \forall T : X \to X\, \big(T \text{ n.e.} \to \exists k \in \mathbb{N}\, A_\exists\big),$$

then one can extract a **computable function** $\Phi : \mathbb{N}^2 \to \mathbb{N}$

s.t. in all normed spaces X it holds that

$$\forall n, b \in \mathbb{N}\, \forall x \in X\, \forall T : X \to X$$
$$\big(T \text{ n.e.} \wedge \|x\|, \|T(0)\| \leq b \to \exists k \leq \Phi(n, b)\, A_\exists\big).$$

Similar for Hilbert spaces and uniformly convex spaces (then bound depends on modulus of convexity). In metric setting: $d(x, Tx) \leq b$.

Special case of **general logical metatheorems** (T nonexpansive):

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, \|\cdot\|]$ proves ('n.e.' means 'nonexpansive')

$$\forall n \in \mathbb{N}\, \forall x \in X\, \forall T : X \to X\, \big(T \text{ n.e.} \to \exists k \in \mathbb{N}\, A_\exists\big),$$

then one can extract a **computable function** $\Phi : \mathbb{N}^2 \to \mathbb{N}$

s.t. in all normed spaces X it holds that

$$\forall n, b \in \mathbb{N}\, \forall x \in X\, \forall T : X \to X$$
$$\big(T \text{ n.e.} \wedge \|x\|, \|T(0)\| \le b \to \exists k \le \Phi(n, b)\, A_\exists\big).$$

Similar for Hilbert spaces and uniformly convex spaces (then bound depends on modulus of convexity). In metric setting: $d(x, Tx) \le b$.

Method: Novel forms of Gödel's functional interpretation!

Applicability of Metatheorem

- Applied to asymptotic regularity statements $d(x_n, Tx_n) \to 0$, the corollary often gives full rates of convergence, e.g. because $(d(x_n, Tx_n))$ is nonincreasing so that $d(x_n, Tx_n) \to 0 \in \forall\exists$.

Applicability of Metatheorem

- Applied to asymptotic regularity statements $d(x_n, Tx_n) \to 0$, the corollary often gives full rates of convergence, e.g. because $(d(x_n, Tx_n))$ is nonincreasing so that $d(x_n, Tx_n) \to 0 \in \forall\exists$.
- From proofs of the convergence of (x_n) itself, one may only get **rates of metastability** Φ (Kreisel 1951, K.05, Tao 07) s.t.

$$\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n, n+g(n)]\, (d(x_i, x_j) < 2^{-k}) \in \forall\exists.$$

- **Admissible abstract structures:** metric, hyperbolic, CAT(0), CAT($\kappa > 0$), Ptolemy, normed, their completions, Hilbert, uniformly convex, uniformly smooth (not: separable, strictly convex or smooth) spaces, abstract L^p- and $C(K)$-spaces (and all other normed structures axiomatizable in positive bounded logic (in the sense of Henson, Iovino, Ben-Yaacov etc.).

- **Admissible abstract structures:** metric, hyperbolic, CAT(0), CAT($\kappa > 0$), Ptolemy, normed, their completions, Hilbert, uniformly convex, uniformly smooth (not: separable, strictly convex or smooth) spaces, abstract L^p- and $C(K)$-spaces (and all other normed structures axiomatizable in positive bounded logic (in the sense of Henson, Iovino, Ben-Yaacov etc.).
- **Admissible classes of functions**: uniformly continuous, Lipschitzian, nonexpansive, firmly- and strongly nonexpansive functions; also some classes of **discontinuous** functions: pseudo-contractions, maps with Suzuki's condition (E) etc.

- **Admissible abstract structures:** metric, hyperbolic, CAT(0), CAT($\kappa > 0$), Ptolemy, normed, their completions, Hilbert, uniformly convex, uniformly smooth (not: separable, strictly convex or smooth) spaces, abstract L^p- and $C(K)$-spaces (and all other normed structures axiomatizable in positive bounded logic (in the sense of Henson, Iovino, Ben-Yaacov etc.).
- **Admissible classes of functions**: uniformly continuous, Lipschitzian, nonexpansive, firmly- and strongly nonexpansive functions; also some classes of **discontinuous** functions: pseudo-contractions, maps with Suzuki's condition (E) etc. **Recently:** set-valued accretive operators (Cauchy problems). (K./Koutsoukou-Argyraki, K./Powell).

- Uses of **ultraproducts** made in model theory can often be replaced by a **proof-theoretic uniform boundedness principle UB** which can be **eliminated** from proofs without contributing to the extracted bounds (K. ENTCS 2006, Engracia 2009, Günzel/K. Adv. Math. 2016). Recently **UB** has been used to replace sequential weak compactness (Ferreira, Leuştean, Pinto, Adv. Math. to appear).

Experience from numerous case studies

- Essentially, all **rates of asymptotic regularity** extracted have a very low complexity (**polynomial or exponential** in the data) at least in the most relevant special cases.

Experience from numerous case studies

- Essentially, all **rates of asymptotic regularity** extracted have a very low complexity (**polynomial or exponential** in the data) at least in the most relevant special cases.
- Except for 2 cases, all **rates of metastability** are of essentially the form

$$\Phi(\underline{a}, g) = (\chi_1(\underline{a}) \circ g \circ \chi_2(\underline{a}))^{B(\underline{a})}(0)$$

for simple (essentially polynomial) functions χ_1, χ_2, B in majorants $\underline{a}$ of the parameters of the problem.

Experience from numerous case studies

- Essentially, all **rates of asymptotic regularity** extracted have a very low complexity (**polynomial or exponential** in the data) at least in the most relevant special cases.
- Except for 2 cases, all **rates of metastability** are of essentially the form

$$\Phi(\underline{a}, g) = (\chi_1(\underline{a}) \circ g \circ \chi_2(\underline{a}))^{B(\underline{a})}(0)$$

for simple (essentially polynomial) functions χ_1, χ_2, B in majorants $\underline{a}$ of the parameters of the problem.
Implies: **algorithmic learnability** of a rate of convergence which - if a gap condition is satisfied - yields **oscillation bounds** (K./Safarik APAL 2014, Avigad/Rute ETDS 2015).

Proof-theoretic versus model-theoretic tameness

- In the recent book 'Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism', John Baldwin has argued that model theory became successful in applications to core mathematics by focusing on **local** foundations/formalizations rather than **global** ones and on **tame structures** (e.g. o-minimal ones).

Proof-theoretic versus model-theoretic tameness

- In the recent book ‘Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism’, John Baldwin has argued that model theory became successful in applications to core mathematics by focusing on **local** foundations/formalizations rather than **global** ones and on **tame structures** (e.g. o-minimal ones).
- We argue, that in a related way, also ‘proof mining’ is successful by focusing on **specific classes of problems** (e.g. iterations of nonlinear operators $T : C \to C$ on general convex subsets of abstract classes of normed or geodesic spaces).

Proof-theoretic versus model-theoretic tameness

- In contrast to model-theoretic tameness, quantification over $\mathbb{N}$ and inductions etc. are crucially used in connection with convergence statements so that **Gödel-phenomena could occur in principle**.

Proof-theoretic versus model-theoretic tameness

- In contrast to model-theoretic tameness, quantification over $\mathbb{N}$ and inductions etc. are crucially used in connection with convergence statements so that **Gödel-phenomena could occur in principle**.
- A different form of **'proof-theoretic tameness'** in existing ordinary (nonlinear) analysis largely leads **to extractable bounds of very low complexity**.

Proof-theoretic versus model-theoretic tameness

- In contrast to model-theoretic tameness, quantification over $\mathbb{N}$ and inductions etc. are crucially used in connection with convergence statements so that **Gödel-phenomena could occur in principle**.
- A different form of **'proof-theoretic tameness'** in existing ordinary (nonlinear) analysis largely leads **to extractable bounds of very low complexity**.
- **Geometric properties** such as uniform convexity and smoothness etc. **more important than complicated inductions**.

Proof-theoretic versus model-theoretic tameness

- To **detect** **proof-theoretic tameness** **requires** to actually carry out the **proof analysis** (though usually some rough upper bound on the complexity can be obtained from proof-theoretic conservation results).

Proof-theoretic tameness in practice I:

Polynomial rate of asymptotic regularity in Bauschke's solution of the 'zero displacement conjecture'

Consider a Hilbert space H and nonempty closed and convex subsets $C_1, \ldots, C_N \subseteq H$ with metric projections P_{C_i}, define $T := P_{C_N} \circ \ldots \circ P_{C_1}$. In 2003 Bauschke proved the 'zero displacement conjecture':

$$\|T^{n+1}x - T^n x\| \to 0 \quad (x \in H).$$

Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^N C_i \neq \emptyset$) or C_i half spaces etc. starting with von Neumann.

Consider a Hilbert space $\boldsymbol{H}$ and nonempty closed and convex subsets $\boldsymbol{C_1, \ldots, C_N \subseteq H}$ with metric projections $\boldsymbol{P_{C_i}}$, define $\boldsymbol{T := P_{C_N} \circ \ldots \circ P_{C_1}}$. In 2003 Bauschke proved the 'zero displacement conjecture':

$$\boldsymbol{\| T^{n+1}x - T^n x\| \to 0 \quad (x \in H).}$$

Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^N C_i \neq \emptyset$) or C_i half spaces etc. starting with von Neumann. Proof uses abstract theory of maximal monotone operators: Minty's theorem, Brézis-Haraux theorem, Rockafellar's maximal monotonicity and sum theorems, strongly nonexpansive mappings, conjugate functions, normal cone operator...).

$(\|T^{n+1}x - T^n x\|)_{n\in\mathbb{N}}$ is **nonincreasing** and hence the conclusion in Bauschke's theorem is of the form $\forall\exists$.

$(\|T^{n+1}x - T^n x\|)_{n\in\mathbb{N}}$ is **nonincreasing** and hence the conclusion in Bauschke's theorem is of the form $\forall\exists$.

Extractability of a uniform rate of asymptotic regularity which only depends on the error $\varepsilon > 0$, $N \in \mathbb{N}$, $b \geq \|x\|$ and $K \geq \|c_1\|, \ldots, \|c_N\|$ for some **arbitrary** points $c_1 \in C_1, \ldots, c_N \in C_N$ since $\|P_{C_i}0\| \leq \|c_i\| \leq K$ and P_{C_i} nonexpansive!

$(\|T^{n+1}x - T^n x\|)_{n\in\mathbb{N}}$ is **nonincreasing** and hence the conclusion in Bauschke's theorem is of the form $\forall\exists$.

Extractability of a uniform rate of asymptotic regularity which only depends on the error $\varepsilon > 0$, $N \in \mathbb{N}$, $b \geq \|x\|$ and $K \geq \|c_1\|, \ldots, \|c_N\|$ for some **arbitrary** points $c_1 \in C_1, \ldots, c_N \in C_N$ since $\|P_{C_i}0\| \leq \|c_i\| \leq K$ and P_{C_i} nonexpansive!

So corollary guarantees a computable $\Phi(\varepsilon, N, b, K)$ s.t.

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(\varepsilon, N, b, K) \; (\|T^{n+1}x - T^n x\| < \varepsilon).$$

Theorem (K. FoCM 2019)

$$\Phi(\varepsilon, N, b, K) := \left\lceil \frac{18b + 12\alpha(\varepsilon/6))}{\varepsilon} - 1 \right\rceil \left\lceil \left(\frac{D}{\omega(D, \tilde{\varepsilon})} \right) \right\rceil$$

is a **rate of asymptotic regularity** in Bauschke's result, where

$$\tilde{\varepsilon} := \frac{\varepsilon^2}{27b + 18\alpha(\varepsilon/6)}, D := 2b + NK, \omega(D, \tilde{\varepsilon}) := \frac{1}{16D}(\tilde{\varepsilon}/N)^2.$$

$$\alpha(\varepsilon) := \frac{(K^2 + N^3(N-1)^2K^2)N^2}{\varepsilon}.$$

Here $b \geq \|x\|$ and $K \geq \left(\sum_{i=1}^N \|c_i\|^2\right)^{\frac{1}{2}}$ for some $(c_1, \ldots, c_N) \in C_1 \times \ldots \times C_N$.

Proof-theoretic tameness in practice II: Pursuit-evasion games: Lion-Man

Let (X, d) be a uniquely geodesic space, $D > 0$. $L_0, M_0 \in A$ starting points of the lion L and the man M. After n-steps, M moves to any point M_n s.t. $d(M_n, M_{n+1}) \leq D$ and L moves via the geodesic $[L_n, M_n]$ s.t. $d(L_n, L_{n+1}) = \min\{D, d(L_n, M_n)\}$.

Let (X, d) be a uniquely geodesic space, $D > 0$. $L_0, M_0 \in A$ starting points of the lion L and the man M. After n-steps, M moves to any point M_n s.t. $d(M_n, M_{n+1}) \leq D$ and L moves via the geodesic $[L_n, M_n]$ s.t. $d(L_n, L_{n+1}) = \min\{D, d(L_n, M_n)\}$.

Lopéz-Acedo/Nicolae/Piatek, Geom.Dedicat. to appear: if X is a compact uniquely geodesic space with the betweenness property, then **the lion wins** i.e. $\mathbf{\lim d(L_{n+1}, M_n) = 0}$ (proof makes iterated use of sequential compactness, i.e. arithmetic comprehension **ACA**).

Let (X, d) be a uniquely geodesic space, $D > 0$. $L_0, M_0 \in A$ starting points of the lion L and the man M. After n-steps, M moves to any point M_n s.t. $d(M_n, M_{n+1}) \leq D$ and L moves via the geodesic $[L_n, M_n]$ s.t. $d(L_n, L_{n+1}) = \min\{D, d(L_n, M_n)\}$.

Lopéz-Acedo/Nicolae/Piatek, Geom.Dedicat. to appear: if X is a compact uniquely geodesic space with the betweenness property, then **the lion wins** i.e. $\lim d(L_{n+1}, M_n) = 0$ (proof makes iterated use of sequential compactness, i.e. arithmetic comprehension **ACA**).

'$\lim d(L_{n+1}, M_n) = 0$'$\in \Pi^0_2$ since the sequence is nonincreasing!

- **Proof Mining extracts an explicit rate of convergence** if one upgrades 'uniquely geodesic' and 'betweenness property' to **'uniform uniquely geodesic (with modulus)'** and **'uniform betweenness property (with modulus Θ)'**.

- **Proof Mining extracts an explicit rate of convergence** if one upgrades 'uniquely geodesic' and 'betweenness property' to **'uniform uniquely geodesic (with modulus)'** and **'uniform betweenness property (with modulus Θ)'**.
- With these upgrades the assumption of **compactness can be replaced by boundedness** in this particular case!

- **Proof Mining extracts an explicit rate of convergence** if one upgrades 'uniquely geodesic' and 'betweenness property' to **'uniform uniquely geodesic (with modulus)'** and **'uniform betweenness property (with modulus Θ)'**.
- With these upgrades the assumption of **compactness can be replaced by boundedness** in this particular case!
- Even the **uniqueness of geodesics can be dropped**.

- **Proof Mining extracts an explicit rate of convergence** if one upgrades 'uniquely geodesic' and 'betweenness property' to **'uniform uniquely geodesic (with modulus)'** and **'uniform betweenness property (with modulus Θ)'**.
- With these upgrades the assumption of **compactness can be replaced by boundedness** in this particular case!
- Even the **uniqueness of geodesics can be dropped**.
- Proof mining provides an explicit **rate of convergence** which only depends on Θ (in addition to $b \geq \text{diam}(A), D, \varepsilon > 0$).

- **Proof Mining extracts an explicit rate of convergence** if one upgrades 'uniquely geodesic' and 'betweenness property' to **'uniform uniquely geodesic (with modulus)'** and **'uniform betweenness property (with modulus Θ)'**.
- With these upgrades the assumption of **compactness can be replaced by boundedness** in this particular case!
- Even the **uniqueness of geodesics can be dropped**.
- Proof mining provides an explicit **rate of convergence** which only depends on Θ (in addition to $b \geq \mathrm{diam}(A), D, \varepsilon > 0$).
- **Moduli of uniform betweenness** can be **extracted** from **proofs of mere betweenness** for the admissible structures.

Betweenness and uniform betweenness in metric spaces

Definition (Diminnie and White 1981)

Let (X, d) be a metric space. X satisfies the betweenness property if for any distinct points $x, y, z, w \in X$

$$\left.\begin{array}{l} d(x,y) + d(y,z) \le d(x,z) \\ d(y,z) + d(z,w) \le d(y,w) \end{array}\right\} \Rightarrow d(x,z) + d(z,w) \le d(x,w).$$

Betweenness and uniform betweenness in metric spaces

Definition (Diminnie and White 1981)

Let (X, d) be a metric space. X satisfies the betweenness property if for any distinct points $x, y, z, w \in X$

$$\left.\begin{array}{l} d(x,y) + d(y,z) \leq d(x,z) \\ d(y,z) + d(z,w) \leq d(y,w) \end{array}\right\} \Rightarrow \; d(x,z) + d(z,w) \leq d(x,w).$$

For normed spaces, betweenness follows from (but is strictly weaker than) strict convexity. It fails for $(\mathbb{R}^2, \|\cdot\|_\infty), (\mathbb{R}^2, \|\cdot\|_1)$ but holds for some nonstrictly convex spaces.

The functional interpretation upgrades betweenness to (equivalent in the compact case!):

Definition (K., Lopéz-Acedo, Nicolae 2019)

A metric space (X, d) satisfies the uniform betweenness property with modulus $\Theta : (0, \infty)^3 \to (0, \infty)$ if

$$\forall \varepsilon, a, b > 0\, \forall x, y, z, w \in X$$
$$\left(\left\{ \begin{array}{l} \mathrm{sep}\{x, y, z, w\} \geq a \wedge \mathrm{diam}\{x, y, z, w\} \leq b \\ d(x, y) + d(y, z) \leq d(x, z) + \Theta(\varepsilon, a, b) \\ d(y, z) + d(z, w) \leq d(y, w) + \Theta(\varepsilon, a, b) \end{array} \right\} \atop \Rightarrow d(x, z) + d(z, w) \leq d(x, w) + \varepsilon \right).$$

Definition (Lion-Man Game in general metric spaces)

X metric space, $D > 0$, $(M_n), (L_n)$ be sequences in X s.t.

$$d(M_n, M_{n+1}) \leq D, \; d(L_{n+1}, L_n) + d(L_{n+1}, M_n) = d(L_n, M_n),$$

$$d(L_n, L_{n+1}) = \min\{D, d(L_n, M_n)\}.$$

Definition (Lion-Man Game in general metric spaces)

X metric space, $D > 0$, $(M_n), (L_n)$ be sequences in X s.t.

$$d(M_n, M_{n+1}) \leq D, \; d(L_{n+1}, L_n) + d(L_{n+1}, M_n) = d(L_n, M_n),$$

$$d(L_n, L_{n+1}) = \min\{D, d(L_n, M_n)\}.$$

Then $\langle (M_n), (L_n) \rangle$ is a **Lion-Man game** with speed $D > 0$.

Definition (Lion-Man Game in general metric spaces)

X metric space, $D > 0$, $(M_n), (L_n)$ be sequences in X s.t.

$$d(M_n, M_{n+1}) \leq D,\ \ d(L_{n+1}, L_n) + d(L_{n+1}, M_n) = d(L_n, M_n),$$

$$d(L_n, L_{n+1}) = \min\{D, d(L_n, M_n)\}.$$

Then $\langle (M_n), (L_n) \rangle$ is a **Lion-Man game** with speed $D > 0$.

Let X be a **b-bounded** metric space with the uniform betweenness property with modulus Θ satisfying

$$\Theta(\varepsilon) := \Theta(\varepsilon, \varepsilon, b) \leq \varepsilon \quad \text{for all } \varepsilon > 0.$$

For $D > 0$ let $N \in \mathbb{N}$ be s.t. $b + 1 < ND$.

Theorem (K./Lopéz-Acedo/Nicolae 2019)

Let X be a bounded metric space with the uniform betweenness property and $\langle (M_n), (L_n) \rangle$ be a Lion-Man game, speed $D > 0$. Then the Lion approaches the man arbitrarily close.

Theorem (K./Lopéz-Acedo/Nicolae 2019)

Let X be a bounded metric space with the uniform betweenness property and $\langle (M_n), (L_n) \rangle$ be a Lion-Man game, speed $D > 0$. Then the Lion approaches the man arbitrarily close.

Moreover with $b \geq \mathrm{diam}(X)$, Θ, N as above:

$$\forall \varepsilon > 0\, \forall n \geq \Omega_{D,b,\Theta}(\varepsilon)\ (d(L_{n+1}, M_n) < \varepsilon),$$

where

$$\Omega_{D,b,\Theta}(\varepsilon) = N + N \left\lceil \frac{b}{\Theta^{(N)}(\alpha)} \right\rceil$$

with

$$0 < \alpha \leq \min\left\{ \frac{1}{N}, \frac{D}{2}, \frac{\varepsilon}{2} \right\}.$$

Moduli of uniform betweennes

Θ can be **explicitly computed** for L^p $(1 < p < \infty)$ (of order 2 if $1 < p < 2$ and of order p if $2 \leq p < \infty$) and **CAT(κ)-spaces, $\kappa > 0$** (of order 2).

Moduli of uniform betweennes

Θ can be **explicitly computed** for L^p $(1 < p < \infty)$ (of order 2 if $1 < p < 2$ and of order p if $2 \leq p < \infty$) and **CAT(κ)-spaces, $\kappa > 0$** (of order 2).

Low complexity Θ's can also be obtained in a number of **non-uniquely geodesic** normed and metric cases!

A borderline case for proof-theoretic tameness

U. Kohlenbach, A. Sipoș, The finitary content of sunny nonexpansive retractions. arXiv:1812.04940 [math.FA], 2018.

The Browder-Halpern result

Let $C \subseteq H$ be a bounded, closed and convex subset of a Hilbert space H. $T : C \to C$ be nonexpansive, $x_0 \in C$ and $t \in [0,1)$.

$$T_t : C \to C, \quad T_t(x) := tTx + (1-t)x_0$$

is a t-contraction and so has a unique point x_t with $x_t = T_t x_t$.

The Browder-Halpern result

Let $C \subseteq H$ be a bounded, closed and convex subset of a Hilbert space H. $T : C \to C$ be nonexpansive, $x_0 \in C$ and $t \in [0, 1)$.

$$\boldsymbol{T_t : C \to C, \quad T_t(x) := tTx + (1 - t)x_0}$$

is a t-contraction and so has a unique point x_t with $x_t = T_t x_t$.

Theorem (Browder 1967; Halpern 1967)

$p := \lim_{t \to 1} x_t$ exists strongly and $p = P_{Fix(T)}x$, where P denotes the metric projection.

The Browder-Halpern result

Let $C \subseteq H$ be a bounded, closed and convex subset of a Hilbert space H. $T : C \to C$ be nonexpansive, $x_0 \in C$ and $t \in [0, 1)$.

$$\boldsymbol{T_t : C \to C, \quad T_t(x) := tTx + (1-t)x_0}$$

is a t-contraction and so has a unique point x_t with $x_t = T_t x_t$.

Theorem (Browder 1967; Halpern 1967)

$p := \lim_{t \to 1} x_t$ exists strongly and $p = P_{Fix(T)}x$, where P denotes the metric projection.

Even in simple cases on $[0, 1]$ there is in general **no computable rate convergence**. However, a **primitive recursive in the simple form as mentioned above** **rate of metastability** is extracted in (K., Adv. Math. 2011).

S. Reich's Theorem

Even for L^p $(1 < p < \infty, p \neq 2)$ open until the celebrated:

S. Reich's Theorem

Even for L^p $(1 < p < \infty, p \neq 2)$ open until the celebrated:

Theorem (S. Reich, 1980)

In the framework above, if X is a **uniformly smooth Banach space**, then for all $x \in C$ we have that $\lim_{t\to 1} x_t := p$ exists and it is a fixed point of T. Moreover $p = Q_{Fix(T)}x$, where Q is the unique **sunny nonexpansive retraction** onto $Fix(T)$.

S. Reich's Theorem

Even for L^p $(1 < p < \infty, p \neq 2)$ open until the celebrated:

Theorem (S. Reich, 1980)

In the framework above, if X is a **uniformly smooth Banach space**, then for all $x \in C$ we have that $\lim_{t \to 1} x_t := p$ exists and it is a fixed point of T. Moreover $p = Q_{Fix(T)}x$, where Q is the unique **sunny nonexpansive retraction** onto $Fix(T)$.

$Q = P$ iff X is a Hilbert space (Bruck 1974).

S. Reich's Theorem

Even for L^p $(1 < p < \infty, p \neq 2)$ open until the celebrated:

Theorem (S. Reich, 1980)

In the framework above, if X is a **uniformly smooth Banach space**, then for all $x \in C$ we have that $\lim_{t \to 1} x_t := p$ exists and it is a fixed point of T. Moreover $p = Q_{Fix(T)}x$, where Q is the unique **sunny nonexpansive retraction** onto $Fix(T)$.

$Q = P$ iff X is a Hilbert space (Bruck 1974).

The convergence of numerous iterative algorithms in nonlinear analysis is based on Reich's theorem!

Sunny nonexpansive retractions

Let E be a nonempty subset of C and $Q : C \to E$. We call Q a **retraction** if for all $x \in E$, $Qx = x$. If Q is a retraction, we call it **sunny** if for all $x \in C$ and $t \geq 0$, $Q(Qx + t(x - Qx)) = Qx$.

Sunny nonexpansive retractions

Let E be a nonempty subset of C and $Q : C \to E$. We call Q a **retraction** if for all $x \in E$, $Qx = x$. If Q is a retraction, we call it **sunny** if for all $x \in C$ and $t \geq 0$, $Q(Qx + t(x - Qx)) = Qx$.

Proposition (Variational Inequality)

A retraction $Q : C \to E$ is sunny and nonexpansive iff for all $x \in C$ and $y \in E$,

$$\langle x - Qx, j(y - Qx) \rangle \leq 0.$$

Sunny nonexpansive retractions

Let E be a nonempty subset of C and $Q : C \to E$. We call Q a **retraction** if for all $x \in E$, $Qx = x$. If Q is a retraction, we call it **sunny** if for all $x \in C$ and $t \geq 0$, $Q(Qx + t(x - Qx)) = Qx$.

Proposition (Variational Inequality)

A retraction $Q : C \to E$ is sunny and nonexpansive iff for all $x \in C$ and $y \in E$,

$$\langle x - Qx, j(y - Qx) \rangle \leq 0.$$

As a consequence, there is at most one sunny nonexpansive retraction $Q : C \to E$ (Bruck 1973).

Sunny nonexpansive retractions

Let E be a nonempty subset of C and $Q : C \to E$. We call Q a **retraction** if for all $x \in E$, $Qx = x$. If Q is a retraction, we call it **sunny** if for all $x \in C$ and $t \geq 0$, $Q(Qx + t(x - Qx)) = Qx$.

Proposition (Variational Inequality)

A retraction $Q : C \to E$ is sunny and nonexpansive iff for all $x \in C$ and $y \in E$,

$$\langle x - Qx, j(y - Qx) \rangle \leq 0.$$

As a consequence, there is at most one sunny nonexpansive retraction $Q : C \to E$ (Bruck 1973).

The existence of (sunny) nonexpansive retractions onto $Fix(T)$ was first shown by R. Bruck in 1971,1973 using Zorn's lemma.

One key step in the proof

Consider $f : C \to \mathbb{R}_+$ with $f(z) := \limsup_{n\to\infty} \|x_n - z\|$. Let K be the set of minimizers of f. Claim: $K \cap Fix(T) \neq \emptyset$.

One key step in the proof

Consider $f : C \to \mathbb{R}_+$ with $f(z) := \limsup_{n\to\infty} \|x_n - z\|$. Let K be the set of minimizers of f. Claim: $K \cap Fix(T) \neq \emptyset$.

Since f is convex and continuous, C is closed convex bounded nonempty, and X is uniformly smooth, hence reflexive, we have that $K \neq \emptyset$.

One key step in the proof

Consider $f : C \to \mathbb{R}_+$ with $f(z) := \limsup_{n\to\infty} \|x_n - z\|$. Let K be the set of minimizers of f. Claim: $K \cap Fix(T) \neq \emptyset$.

Since f is convex and continuous, C is closed convex bounded nonempty, and X is uniformly smooth, hence reflexive, we have that $K \neq \emptyset$. Let $y \in K$ and $z \in C$. Then:

$$\begin{aligned} f(Ty) = \limsup_{n\to\infty} \|x_n - Ty\| &\leq \limsup_{n\to\infty} (\|x_n - Tx_n\| + \|Tx_n - Ty\|) \\ &\leq \limsup_{n\to\infty} (\|x_n - Tx_n\| + \|x_n - y\|) \\ &\leq \limsup_{n\to\infty} \|x_n - Tx_n\| + \limsup_{n\to\infty} \|x_n - y\| \\ &= f(y) \leq f(z), \end{aligned}$$

so $Ty \in K$.

One key step in the proof

Consider $f : C \to \mathbb{R}_+$ with $f(z) := \limsup_{n\to\infty} \|x_n - z\|$. Let K be the set of minimizers of f. Claim: $K \cap Fix(T) \neq \emptyset$.

Since f is convex and continuous, C is closed convex bounded nonempty, and X is uniformly smooth, hence reflexive, we have that $K \neq \emptyset$. Let $y \in K$ and $z \in C$. Then:

$$\begin{aligned} f(Ty) = \limsup_{n\to\infty} \|x_n - Ty\| &\leq \limsup_{n\to\infty}(\|x_n - Tx_n\| + \|Tx_n - Ty\|) \\ &\leq \limsup_{n\to\infty}(\|x_n - Tx_n\| + \|x_n - y\|) \\ &\leq \limsup_{n\to\infty} \|x_n - Tx_n\| + \limsup_{n\to\infty} \|x_n - y\| \\ &= f(y) \leq f(z), \end{aligned}$$

so $Ty \in K$. Since K is a closed convex bounded nonempty T-invariant subset of a uniformly smooth space, there is a $p \in K \cap Fix(T)$.

The proof gets somewhat easier if X is assumed to be also **uniformly convex** (still covering L^p-spaces): only **arbitrarily good ε-minimizers** needed.

The proof gets somewhat easier if X is assumed to be also **uniformly convex** (still covering L^p-spaces): only **arbitrarily good ε-minimizers** needed.

Next: replace **lim sup**'s ($\Leftrightarrow$ACA) by **approximate lim sup**'s ($\Leftrightarrow \Pi^0_2$-IA), in a process known as **arithmetization** (K.1996).

The proof gets somewhat easier if X is assumed to be also **uniformly convex** (still covering L^p-spaces): only **arbitrarily good ε-minimizers** needed.

Next: replace **lim sup**'s ($\Leftrightarrow$ACA) by **approximate lim sup**'s ($\Leftrightarrow \Pi^0_2$-IA), in a process known as **arithmetization** (K.1996).

While the functional interpretation of the ε-minimization in general requires Gödel primitive recursion of type 2 ($\boldsymbol{T_2}$) the actual extraction constructs a **rate of metastability** in $\boldsymbol{\Phi} \in \boldsymbol{T_1}$ (i.e. of Ackermann type).

The proof gets somewhat easier if X is assumed to be also **uniformly convex** (still covering L^p-spaces): only **arbitrarily good ε-minimizers** needed.

Next: replace **lim sup**'s ($\Leftrightarrow$ACA) by **approximate lim sup**'s ($\Leftrightarrow \Pi^0_2$-IA), in a process known as **arithmetization** (K.1996).

While the functional interpretation of the ε-minimization in general requires Gödel primitive recursion of type 2 ($\boldsymbol{T_2}$) the actual extraction constructs a **rate of metastability** in $\boldsymbol{\Phi} \in \boldsymbol{T_1}$ (i.e. of Ackermann type).

It may well be that a closer analysis of of Φ shows that it is already definable in $\boldsymbol{T_0}$ (in line with a classical result of Parsons that certain forms of type-1 primitive recursion can be reduced to $\boldsymbol{T_0}$).

Future of Proof Mining

- Find new ‘organic connections’ (Kreisel) between core mathematics and proof theory.

Future of Proof Mining

- Find new ‘organic connections’ (Kreisel) between core mathematics and proof theory.
- New classes of results rewarding for proof-theoretic analysis.

Future of Proof Mining

- Find new 'organic connections' (Kreisel) between core mathematics and proof theory.
- New classes of results rewarding for proof-theoretic analysis.
- Proofs which use highly abstract 'ideal' principles to prove concrete numerically meaningful results are most promising.

Future of Proof Mining

- Find new ‘organic connections’ (Kreisel) between core mathematics and proof theory.
- New classes of results rewarding for proof-theoretic analysis.
- Proofs which use highly abstract ‘ideal’ principles to prove concrete numerically meaningful results are most promising.
- Built suitable local proof-theoretic methods to cover such classes of proofs appropriately.

Future of Proof Mining

- Find new 'organic connections' (Kreisel) between core mathematics and proof theory.
- New classes of results rewarding for proof-theoretic analysis.
- Proofs which use highly abstract 'ideal' principles to prove concrete numerically meaningful results are most promising.
- Built suitable local proof-theoretic methods to cover such classes of proofs appropriately.
- The area of analysis has been particularly fruitful. But other promising areas: geometry, algebra (see Simmons/Towsner Adv.Math.).

Recent Surveys:

Recent Surveys:

U. Kohlenbach, Proof-Theoretic Methods in Nonlinear Analysis. In. Proc. ICM 2018, Proc. ICM 2018, B. Sirakov, P. Ney de Souza, M. Viana (eds.), Vol. 2, pp. 61-82. World Scientific 2019.

Recent Surveys:

U. Kohlenbach, Proof-Theoretic Methods in Nonlinear Analysis. In. Proc. ICM 2018, Proc. ICM 2018, B. Sirakov, P. Ney de Souza, M. Viana (eds.), Vol. 2, pp. 61-82. World Scientific 2019.

U. Kohlenbach, Local Formalizations in Nonlinear Analysis and Related Areas and Proof-Theoretic Tameness. To appear in forthcoming volume (eds. P. Weingartner, H.-P. Leeb) 'Kreisel's Interests - On the Foundations of Logic and Mathematics'.