

Frege's attitude towards sets

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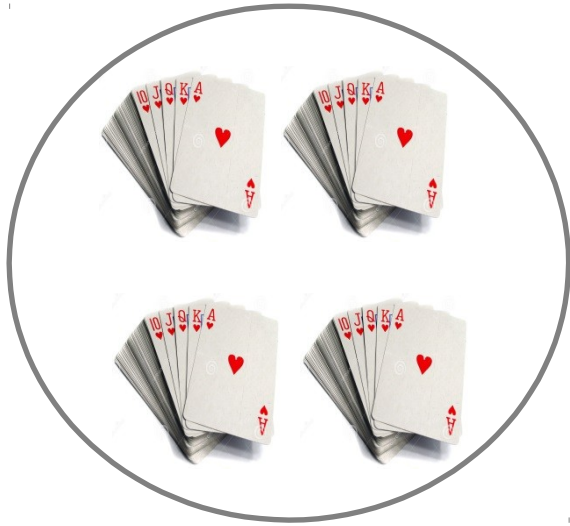
Georg Cantor

“By a ‘manifold’ (Mannigfaltigkeit) or a ‘set’ (Menge) I namely understand generally every multiplicity (Viele) which can be thought as one, i.e. every complex (Inbegriff) of definite elements which can be united to a whole by a law, and by this I believe I have defined something that is related to the Platonic eidos or idea...”

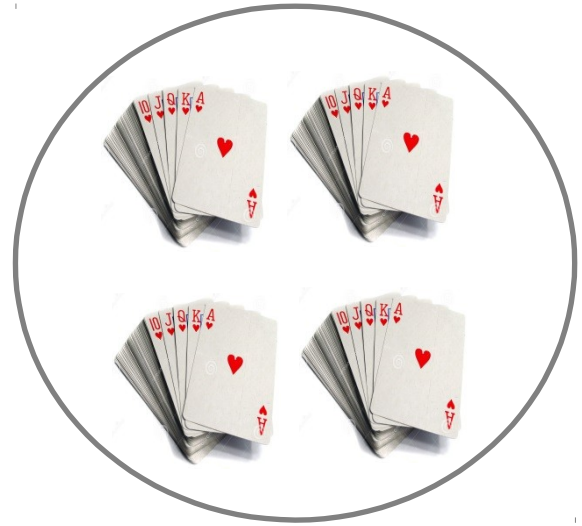
Cantor, Grundlagen einer Allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre der Unendlichen, 1883

“By a ‘set’ (Menge) we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought. These objects are called the ‘elements’ of M .”

Cantor: Beiträge zur Begründung der transfiniten Mengenlehre, 1895



≠



„The fact that classes are universals, or abstract entities, is sometimes obscured by speaking of classes as mere aggregates or collections, thus likening a class of stones, say, to a heap of stones. The heap is indeed a concrete object, as concrete as the stones that make it up; but the class of stones in the heap cannot properly be identified with the heap. For, if it could, then by the same token another class could be identified with the same heap, namely, the class of molecules of stones in the heap. But actually these classes have to be kept distinct; for we want to say that the one has just, say, a hundred members, while the other has trillions. **Classes, therefore, are abstract entities;** we may call them aggregates or collections if we like, but **they are universals.** That is, if there are classes.“

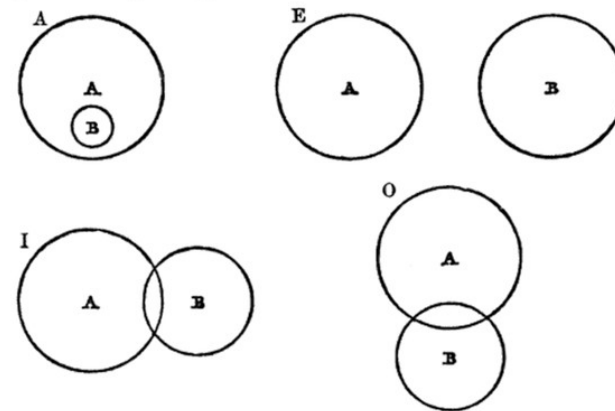
Quine, From a Logical Point of View

Aristotelian syllogistic

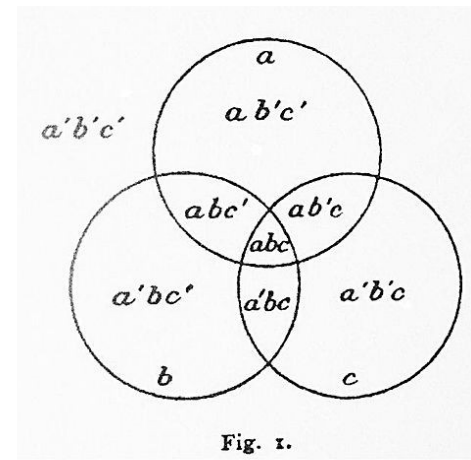
“That one term should be in another as in a whole is the same as for the other to be predicated of all of the first. And we say that one term is predicated of all of another, whenever nothing can be found of which the other term cannot be asserted.”

Aristotle, Prior Analytics [2, 24b27-30]

Euler-Diagram:



Venn-Diagram:



“This use of ‘individual’ has nothing to do with the distinction between ‘individuals’ and ‘classes’ in logic (...). In the logical sense the sets are individuals.”

Bernays, P., Axiomatic Set Theory

„In set theory (...) sets are given axiomatically, so their existence and basic properties are postulated by the appropriate formal axioms.“

Stanford Encyclopedia of Philosophy

Kurt Gödel:

„Mathematical logic, which is nothing else but a precise and complete formulation of formal logic, has two quite different aspects. On the one hand, it is a section of Mathematics treating of classes, relations, combinations of symbols, etc., instead of numbers, functions, geometric figures, etc. On the other hand, it is a science prior to all others, which contains the ideas and principles underlying all sciences.“

Gödel, Russell's Mathematical Logic (1944)

X

Gottlob Frege:

„The class, namely, is something derived, whereas in the concept – as I understand the word – we have something primitive” and “the primitive laws of Logic may contain nothing derived”.

Frege, Philosophical and Mathematical Correspondence

Gottlob Frege

„The fundamental logical relation is that of an object’s falling under a concept: all relations between concepts can be reduced to this.“

Comments on Sense and Meaning

„I call the concepts under which an object falls its **properties**; thus ‚to be Φ is a property of Γ ‘ is just another way of saying: ‚ Γ falls under the concept of a Φ ‘.“

Concept and Object

“The peculiarity of functional signs, which we here called ‘unsaturatedness’, naturally has something answering to it in the functions themselves. They too may be called ‘**unsaturated**’, and in this way we mark them out as fundamentally different from numbers. Of course this is no definition; but likewise none is here possible. I must confine myself to hinting at what I have in mind by means of a metaphorical expression, and here I rely on my reader's agreeing to meet me half way.”

What is a function?

“Criterion, then, takes place according to a law, and different laws of this sort can be thought of. In that case, the expression ‚y is a function of x‘ has no sense, unless it is completed by mentioning the law of correlation. (...) [T]he law (...) is really the main thing. (...) **Distinctions between laws of correlation will go along with distinctions between functions**; and these cannot any longer be regarded as quantitative.“

What is a Function?

„[I]t is precisely by the notation that uses ,x' to indicate a number indefinitely that we are led to the right conception. People call x the argument, and recognize the same function again in ,2 . 1³ + 1', ,2 . 4³ + 4', 2 . 5³ + 5' only with different arguments, viz. 1, 4, and 5. From this we may discern that it is the common element of these expressions that contains the essential peculiarity of a function; i.e. what is present in '2 . x³ + x' over and above the letter 'x'. We could write this somewhat as follows: '2 . ()³ + ()'”

Function and Concept

„For instance, if I say ,the function 2 . x³ + x', x must not be considered as belonging to the function; this letter only serves to **indicate the kind of supplementation that is needed**; it enables one to recognize the places where the sign for the argument must go in.“

Function and Concept

„Now just as functions are fundamentally different from objects, so also functions whose arguments are and must be functions are fundamentally different from functions whose arguments are objects and cannot be anything else. I call the latter first-level, the former second-level, functions.“

Function and Concept

“the behaviour of the concept is essentially predicative, even where something is being said about it; (...) second-level concepts, which concepts fall under, are essentially different from first-level concepts, which objects fall under. (...). **I do not want to say it is false to say concerning an object what is said (...) concerning a concept; I want to say it is impossible, senseless, to do so**”.

Concept and Object

„Frege (...) construed a general term as *naming* its extension - the class of all things of which the term is true - and he construed a statement as naming its truth value.“

Quine, Semantics and Abstract Objects

„For example, the sense of the phrase ‚the number 2‘ does not hold together with that of the expression ‚the concept *prime number*‘ without a link. We apply such a link in the sentence ‚the number 2 falls under the concept *prime number*‘; it is contained in the words ‚falls under‘, which need to be completed in two ways – by a subject and an accusative; and only because their sense is thus ‚unsaturated‘ are they capable of serving as a link. Only when they have been supplemented in this twofold respect do we get a complete sense, a thought. I say that what such words or phrases mean is a relation. We now get the same difficulty for the relation that we were trying to avoid for the concept. For the words ‚the relation of an object to the concept it falls under‘ designate not a relation but an object; and the three proper names ‚the number 2‘, ‚the concept *prime number*‘, ‚the relation of an object to a concept it falls under‘, hold aloof from one another just as much as the first two do by themselves; however we put them together, we get no sentence. It is thus easy for us to see that the difficulty arising from the ‚unsaturatedness‘ of one part of the thought can indeed be shifted, but not avoided.“

Concept and Object

„concepts can have identical extensions without themselves coinciding.“

Concept and Object

“I turn first to the paradoxes of set theory. They arise because a concept, e.g. fixed star, is connected with something that is called the set of fixed stars, which appears to be determined by the concept – and determined as an object. I thus think of the objects falling under the concept fixed star combined into a whole, which I construe as an object and designate by a proper name, ‘the set of fixed stars’. This transformation of a concept into an object is **inadmissible**; for the set of fixed stars only seems to be an object; in truth there is **no such object at all**.”

“The expressions ‘the extension of F’ seems naturalized by reason of its manifold employment and certified by science, so that one does not think it necessary to examine it more closely; but experience has shown how easily this can get one into a morass. I am among those who have suffered this fate. When I tried to place number theory on scientific foundations, I found such an expression very convenient. While I sometimes had slight doubts during the execution of the work, I paid no attention to them. And so it happened that after the completion of the *Basic Laws of Arithmetic* the whole edifice collapsed around me. Such an event should be a warning not only to oneself but also to others. We must set up a warning sign visible from afar: let no one imagine that he can transform a concept into an object.”

“In my fashion of regarding concepts as functions, we can treat the principal parts of Logic without speaking of classes, as I have done in my *Begriffsschrift*, and that difficulty does not then come into consideration. (...) The difficulties which are bound up with the use of classes vanish if we only deal with objects, concepts and relations, and this is possible in the fundamental part of Logic.”

Wissenschaftlicher Briefwechsel

“Set theory in ruins. My concept-script in the main not dependent on it.”

On Schoenflies: Die logischen Paradoxien der Mengenlehre

Axioms of *Begriffsschrift* (Concept Script):

$$A \rightarrow (B \rightarrow A)$$

$$(C \rightarrow (B \rightarrow A)) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow A))$$

$$(D \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow (D \rightarrow A))$$

$$(B \rightarrow A) \rightarrow (\neg A \rightarrow \neg B)$$

$$\neg \neg A \rightarrow A$$

$$A \rightarrow \neg \neg A$$

$$c \equiv d \rightarrow (f(c) \rightarrow f(d))$$

$$c \equiv c$$

$$\forall x f(x) \rightarrow f(c)$$

modus ponens, rule of generalisation, rule of substitution

To sum up:

- Sets are not needed in foundations of classical logic.
- Sets are not good substitutes for properties and relations; they are derived, not fine-grained enough, cannot link.
- The notion of sets as objects is contradictorical.

Thank you for your attention!

„Its [of Begriffsschrift] only flaw is some confusion about quantification over functions. Frege reluctantly accepted such quantification because it is needed in his logical definition of sequence, hence of natural number (when the 'ancestral' of a relation is introduced).”

van Heijenoort: Historical Development of Modern Logic

„Frege allows a functional letter to occur in a quantifier (...). This license is not a necessary feature of quantification theory, but Frege has to admit it in his system for the definitions and derivations of the third part of the book.”

van Heijenoort: Introduction to Frege's Begriffsschrift

„Unter einer ‚Mannigfaltigkeit‘ oder ‚Menge‘ verstehe ich nämlich allgemein jedes Viele, welches sich als Einem denken lässt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann, und ich glaube hiermit etwas zu definieren, was verwandt ist mit dem platonischen *eidos* oder *idea* (...)“

„Unter einer ‚Menge‘ verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die ‚Elemente‘ von M genannt werden) zu einem Ganzen.“

“We can say that a class is any aggregate, any collection, any combination of objects of any sort; if this helps, well and good. But even this will be less help than hindrance unless we keep clearly in mind that the aggregating or collecting or combining here is to connote no actual displacement of the objects, and further that the aggregation or collection or combination of say seven given pairs of shoes is not to be identified with the aggregation or collection or combination of those fourteen shoes, nor with that of the twenty-eight soles and uppers. In short, a class may be thought of as an aggregate or collection or combination of objects just so long as ‘aggregate’ or ‘collection’ or ‘combination’ is understood strictly in the sense of ‘class’.”

Quine, W. V., Set Theory and Its Logic

George Boole

“By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the terms ‘nothing’ and ‘universe,’ which as ‘classes’ should be understood to comprise respectively ‘no beings,’ ‘all beings.’”

Boole, G., *An Investigation of the Laws of Thought*, 1854