Precipitous Ideals, Generic Ultrapowers, and Destroying Saturation

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Logic Colloquium 2019

Context

1. Forcing
2. Class Ultrapowers

Class Ultrapowers

1. $U \subseteq \mathcal{P}^V(A)$ a V-ultrafilter on A, i.e.
   - U a filter on A
   - If $B \in V$, $B \subseteq A$, then $B \in U$ or $A \setminus B \in U$

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2. $Ult(V, U) = \{[f] \mid f : A \to V,\ f \in V\}$ by:
   $[f] \in^* [g]$ iff $\{a \in A \mid f(a) \in g(a)\} \in U$
   $[f] = [g]$ iff $\{a \in A \mid f(a) = g(a)\} \in U$

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   $[f] = [g]$ iff $\{a \in A \mid f(a) = g(a)\} \in U$
3. $j : V \to Ult(V, U)$, $j(x) = [a \mapsto x]$ is elementary
   Will often conflate $Ult(V, U)$ with $M = trcoll(Ult(V, U))$ when wellfounded

Table of Contents

Definition

I *ideal* on $\mathcal{P}(\kappa)$ is $I \subseteq \kappa$ with

1. $\emptyset \in I$, $\kappa \notin I$
2. $A \in I$, $B \subseteq A \implies B \in I$
3. $A, B \in I \implies A \cup B \in I$

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Further, I is

- κ-complete if $\tau < \kappa$, $\langle N_\alpha \mid \alpha < \tau \rangle \subseteq I \implies \bigcup_{\alpha<\tau} N_\alpha \in I$
- normal if closed under *diagonal unions*:
  if $\langle N_\alpha \mid \alpha < \kappa \rangle \subseteq I$, then
  $\nabla_{\alpha<\kappa} N_\alpha = \{\beta < \kappa \mid \exists \alpha < \beta \; \beta \in N_\alpha\} \in I$
  equivalently, $f : \kappa \to \kappa$, $f(\alpha) < \alpha$ I^+-often $\implies$ f constant I^+-often

Example

for $\kappa \geq \omega$: $I_{fin} := \{A \subseteq \kappa \mid |A| < \aleph_0\}$ ideal

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Example

for $\kappa \geq \omega_1$: NS_κ κ-complete normal ideal

Definition

I ideal on κ, A, $B \subseteq \kappa$:

$A \leq_I B$ if $A \setminus B \in I$

$A \equiv_I B$ if $A \leq_I B$ and $B \leq_I A$.

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I ideal on κ, A, $B \subseteq \kappa$:
$A \leq_I B$ if $A \setminus B \in I$
$A \equiv_I B$ if $A \leq_I B$ and $B \leq_I A$.
Then $\mathcal{P}(\kappa)/I := \{[A]_{\equiv_I} \mid A \subseteq \kappa\}$, $<_I$ induced by $\leq_I$.

Definition

I ideal on κ is λ-*saturated* if
for any $\langle S_\alpha \mid \alpha < \lambda \rangle$ family of I^+-sets
there are $\alpha < \beta$ such that $S_\alpha \cap S_\beta \in I^+$.

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Theorem

For I κ-complete and κ-saturated:
$\mathbb{P} := (\mathcal{P}(\kappa)/I, <_I) \setminus [\emptyset]$ *is a complete Boolean algebra with κ-cc,*
(i.e. no antichains of size $\geq \kappa$)
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Given G $\mathbb{P}$-generic over V: $U = \{A \in \mathcal{P}^V(\kappa) \mid [A] \in G\}$ is a V-ultrafilter that is:

- *V-κ-complete: $\lambda < \kappa$, $\langle B_\alpha \mid \alpha < \lambda \rangle \in V$, each $B_\alpha \in U \implies \bigcap_{\alpha<\lambda} B_\alpha \in U$*
- *V-normal: $f : \kappa \to \kappa$, $f \in V$, $f(\alpha) < \alpha$ U-often $\implies f$ constant U-often*

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$Ult(V, U)$ is well-founded and $j : V \to Ult(V, U)$, $crit(j) = \kappa$.

Remark

Often, we write $\mathcal{P}(\kappa)/I$ when we mean the forcing $\mathcal{P}(\kappa)/I \setminus [\emptyset]$.

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Definition
I is λ-*presaturated* if I is precipitous and $\mathcal{P}(\kappa)/I$ preserves λ.

Theorem
I λ-presaturated iff for every $\mu < \lambda$, $\langle A_\alpha \mid \alpha < \mu \rangle$ antichains of $\mathcal{P}(\kappa)/I$,
for densely many $[X]$, for all α, $[X]$ compatible with $< \mu$-many $[Y] \in A_\alpha$.

Theorem
For κ a regular cardinal, κ^+-saturated implies κ^+-presaturated implies precipitous.

Table of Contents

Question (Foreman)

For J ideal on $\mathcal{P}(\kappa^{+n+1})$*,*
Let I projected ideal to $\mathcal{P}(\kappa^{+n})$*, i.e.* $I = \{N \cap \kappa^{+n} \mid N \in J\}$
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Suppose canonical homomorphism $\mathcal{P}(\kappa^{+n})/I \to \mathcal{P}(\kappa^{+n+1})/J$ regular
Then: Is I κ^{+n+1}-saturated?

Theorem (Cox, Eskew, Zeman)

No. For counterexample, suffices to find some I κ^{+n+1}-presaturated, non-κ^{+n+1}-saturated ideal on κ^{+n}.

Theorem (Cox, Eskew)

Such ideals consistently exist for $n = 0$ and κ successor of singular

Theorem

Same with $n = 0$ and $\kappa = \mu^+$ regular, with $\mu^{<\mu} = \mu$, forcable by a specific poset $\mathbb{P}(\mu, \kappa)$

Question

What about κ inaccessible?

Table of Contents

Theorem (Cox, Eskew 2019)

For μ regular, $\mu^{<\mu} = \mu$, and $\kappa = \mu^+$.
Let $\mathbb{P}(\mu, \kappa)$ force a club through κ with $< \mu$-conditions.
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- *For I κ-complete, κ^+-saturated in V,* $V^{\mathbb{P}(\mu,\kappa)} \models \bar{I} = \{A \in \mathcal{P}^{V[G]}(\kappa) \mid \exists N \in I \ A \subseteq N\}$ *is κ^+-presaturated*

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Remark

$\mathbb{P}(\omega_1, \omega_2)$: Baumgartner-Taylor (1982), forces club of ω_2 with finite conditions

Theorem (Sinapova, S.)

Let $V \models GCH$, with κ inaccessible, κ-complete κ^+ saturated ideals on κ.
Let $\mathbb{Q}$ be Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular.

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Towards a (partial) proof: Key properties of $\mathbb{P}(\mu, \mu^+)$

For μ regular, $\mathbb{P}(\mu, \mu^+)$ has:

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4. For every $\theta \geq \mu^{++}$ and $M \prec (H_\theta, \in, \mu^+)$ with $M \cap \mu^+ \in \mu^+ \cap cof(\mu)$, $\mathbb{P}(\mu, \mu^+)$ is strongly proper for M

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5. In $V[G]$, G $\mathbb{P}(\mu, \mu^+)$-generic, is C_μ a club subset of μ^+ with $X \in V$, $|X| \geq \mu \implies X \not\subseteq C_\mu$

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Proposition (Destroy saturation)

If J κ-complete κ^+ saturated in V, then $\overline{J}$ is not κ^+-saturated in $V^{\mathbb{Q}}$.

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Lemma (Baumgartner-Taylor, Laver)

With $j : V \to Ult(V, U)$ generic elementary embedding given by $\mathcal{P}(\kappa)/J$:
$\overline{J}$ is κ^+-saturated in $V^{\mathbb{Q}}$ iff $\Vdash_{\mathcal{P}(\kappa)/J}$ "$\dot{j}(\mathbb{Q})$ is κ^+-cc"

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Proof of proposition.

In $V^{\mathcal{P}(\kappa)/J}$, U a $\mathcal{P}(\kappa)/J$-generic:
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In $Ult(V, U)$, $j(\mathbb{Q})(\kappa) = \mathbb{P}(\kappa, \kappa^+)$ hence $j(\mathbb{Q})$ is not κ^+-cc. □

Towards a (partial) proof: Preserving (some) Presaturation

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Tools we need:

Lemma

If $\mathbb{Q}_1, \ldots, \mathbb{Q}_n$ posets, such that
$\Vdash_{\mathbb{Q}_1 * \cdots * \mathbb{Q}_i} \mathbb{Q}_{i+1}$ λ-cc or λ-directed closed
then $\mathbb{Q}_1 * \cdots * \mathbb{Q}_n$ is λ-presaturated.

Lemma

If $\mathbb{P} * \mathbb{Q}$ λ-presaturated
then $\mathbb{P}$ λ-presaturated and $\Vdash_{\mathbb{P}} \mathbb{Q}$ λ-presaturated.

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Lemma (Foreman's Duality Theorem)

For J a κ-complete normal precipitous ideal in V
$\mathbb{P}$ a κ-cc poset in V:
Then

$$\mathcal{B}\left(\mathbb{P} * \mathcal{P}(\kappa)/\overline{J}\right) \cong \mathcal{B}\left(\mathcal{P}(\kappa)/J * \dot{j}\,(\mathbb{P})\right)$$

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For $J \in V$ a κ-complete normal κ^+-saturated ideal:
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Proof.

For U a $\mathcal{P}(\kappa)/J$-generic filter: $\dot{j}(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{R}}$,
$\dot{\mathbb{R}}$ Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, over $\mu \in [\kappa^+, j(\kappa))$ regular in $Ult(V, U)$.

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By Duality, $\mathbb{Q} * \mathcal{P}(\kappa)/\overline{J}$ is κ^+-presaturated

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Hence in $V^{\mathbb{Q}}$, $\overline{J}$ is κ^+-presaturated. □

Question (Cox, Eskew 2019)

Can the above be done at the successor of a singular?

Question

Is the use of $GCH_{<\kappa}$ necessary?

Question

Does $\mathbb{Q}$ preserve κ^+-presaturation of all κ^+-presaturated ideals?