On Gaisi Takeuti's Philosophy of Mathematics

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Aim of this Talk

- Explaining Takeuti's finitism.
- Two focuses:
  - To explain Takeuti's program
    - Difference from Hilbert's program
  - Connection with Nishida's philosophy in Japan.
- Motivation:
  - Who was Takeuti?
    - He has been chiefly known as a great mathematician,
    - But, he wrote many papers in English and Japanese where he expressed his philosophical thoughts.
- Goals:
  - To describe Takeuti's program in a proper context.
  - To write a book about Takeuti.
- This is joint work with Andrew Arana (Université Paris 1 - Panthéon-Sorbonne)
  - Any comments are very welcome!

Gaisi Takeuti

- Brief biography:
  - He was born in Kanazawa in 1926 and graduated from Tokyo Uni. in 1947.
  - He became a professor at the University of Illinois at Urbana-Champaign until 1996.
  - He passed away in 2017.
- He is well-known in proof-theory, set-theory, and quantum logic...

Takeuti's Early Works in 1940's-1960's

- Takeuti's goal: to clarify the notion of set.

  (His first paper on this subject: "On the foundations of mathematics" (**「数学の基礎について」** in Japanese), 1949.

- Formulation of ***GLC*** and Takeuti's conjecture: "On a generalized logic calculus", 1953.
- Invention of new ordinals: "Ordinal diagrams", 1957.
- Consistency proof of Π^1_1-CA_0 "On the fundamental conjecture of ***GLC V***", 1958.
- Consistency proof of Π^1_1-CA+BI: "Consistency proofs of some subsystems of analysis", 1967.

Takeuti's Conjecture

- Takeuti's conjecture: the cut-elimination holds for his higher-order logic called ***GLC***.

  Cf. Gaisi Takeuti, "On a generalized logic calculus", 1953.

- G^1LC : second-order logic

$$\frac{\Gamma \vdash \Delta, \forall X A(X)}{\Gamma \vdash \Delta, A(T)}$$

  - $T \equiv \lambda x.B(x)$.
  - $A(T)$: the result of replacing $X(t)$ occurring in $A(X)$ by $B(t)$.

Example

Let $A(X) \equiv X(0)$ and $B(x) \equiv \forall X(X(x))$. Then, $A(T) \equiv \forall X(X(0))$.
No subformula property!

Partial Solutions to Takeuti's Conjecture

- Difficulty of the conjecture for $\boldsymbol{GLC^1}$:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash A(Y)\end{matrix}}{\Gamma \vdash \forall X A(X)}\ \forall R \qquad \dfrac{\begin{matrix}\vdots \\ A(T) \vdash \Delta\end{matrix}}{\forall X A(X) \vdash \Delta}\ \forall L}{\Gamma \vdash \Delta}\ \mathit{Cut} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \Gamma \vdash A(T)\end{matrix} \quad \begin{matrix}\vdots \\ A(T) \vdash \Delta\end{matrix}}{\Gamma \vdash \Delta}\ \mathit{Cut}$$

  $A(T)$ might be much more complicated than $\forall X A(X)$!

- Takeuti's ordinal diagrams: an abstract expression of the cut-elimination steps:
  - Define the reduction relation $r \hookrightarrow r'$ and the ordinal $ord(r)$ such that, if $r \hookrightarrow r'$, then

$$ord(r) > ord(r')$$

Cf. "On the fundamental conjecture of GLC V", *J. Math. Soc. Japan*, 1958.

- Discovering an infinitary structure (ordinal) behind a finite one (proof-figure).

Takeuti's Influence on Proof-Theory

- Model theoretic proofs of Takeuti's conjecture (1960's):
  - Second-order case: Tait,
  - Higher-order case: Prawitz and Takahashi.
- Girard's proof of the strong normalization for System $\boldsymbol{F}$ and Π^1_2-logic.

  Cf. Girard's talk at Kobe in 2018.
  http://girard.perso.math.cnrs.fr/Kobe2018.pdf
- Infinitary proof-theory in 1970's-1980's: Tait, Buchholz, Pohlers, and Jäger...
  - Iterations of Π^1_1-CA. Connection with theories of inductive definitions (triggered by Feferman, Kreisel) and KP set theory.

  Cf. Buchholz, Feferman, Pohlers, and Sieg, "Iterated Inductive Definitions and Subsystems of Analysis", LNM 897.
- Proof-theory of Π^1_2-CA by Rathjen and Arai in 1990's.

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Hilbert's Standpoint

- Aim of Hilbert's program: justification of our use of infinity.
- No use of such a notion in a meta-theory.
  - finite operations on objects which are definable in finitistic standpoint.
  - ex. concatenation of proofs in natural ded. Also, math. ind. for quantifier free formulas.
- Epistemic consideration in finitism: reliability (Sicherheit)

Über das Unendliche in 1926

Hilbert's use of the word "reliable":

> *[A]s a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable (sicher), it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated , is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication.*

Takeuti's Explanation about Hilbert

When Takeuti writes about Hilbert, he uses the same word.

> *Only the arguments which are applied to the formalized proofs that are concretely considered as above and which are simple and reliable can be admitted. Namely, the strings which are relevant here are concrete strings consisting of finitely many symbols. This standpoint is called the finite standpoint.*

Cf. "Axioms of Arithmetic and Consistency" (*Memoirs of a Proof Theorist: Gödel and Other Logicians*, 2003)

Takeuti's Explanation about Gentzen

Takeuti's use of another word for explaining Gentzen's work;

> *This proof is very clear and transparent if one is familiar with the primitive recursive structure of the ordinals less than* ε_0*. [...] With the first definition [the recursive definition of natural numbers] we are lead naturally to a conscientious interpretation of quantities and here we become conscious of what the problem of foundations really entails.*

Cf. "Consistency proofs and Ordinals" in 1975.

- Use of the word "conscientious", but not "reliable".
- Natural question: what is Takeuti's view about formalism if he did not aim to confrim the reliability by a proof-theoretic project?

Takeuti's View about the Consistency Proofs

For him, the consistency proof gives an examination of the formalism.

> *Thought-objectification in mathematics via symbol has arrived at proofs in mathematics in a formalism via propositions (sets) and logical concepts. What is the significance of it? Or, will it have a great influence on mathematics similar to the case of set? For answering these questions, it seems that formalism is too immature. This new gateway of the new virgin field is too difficult to open. Is formalism too much a premature baby? For me, the consistency problems seem a touchstone for formalism and the foundation of formalism rather than the foundation of mathematics. Still I believe in a bright future for formalism, though one laden with difficulties.*

Cf. "From the viewpoint of formalism II" **「形式主義の立場から II」**, 1956.

- Note: this paper is published in 1956.
    - Gaisi Takeuti, "Ordinal diagrams", 1957.
    - Gaisi Takeuti, "On the fundamental conjecture of GLC V", 1958.

Summary of Takeuti's View about Formalism

- Takeuti's aim was not to confirm the reliability of infinitary mathematics.
- Rather, he aimed to develop formalism by his proof-theoretic project.
  - Contrast with Hilbertian view about formalism.
- After discussions with Gödel, Takeuti started to explain his thoughts in more detail.

“About Mathematics”: Several “Minds”

- Takeuti’s use of the word “mind” coined by Gödel.
  (after discussions with Gödel (1959-60, 66-68, 71-72))
  Cf. Proof Theory and Set Theory, *Synthese*, 1985.
- Our focus here: “About Mathematics” (『科学基礎論研究』, 1972).

Usually, it is essential for modern mathematics that it supposes an infinite mind and conjectures what it does. By infinite mind, I mean a mind which can investigate infinitely many things by checking one by one. For example, the law of excluded middle holds for an infinite mind because it can check whether $A(x)$ *or* $\neg A(x)$ *one by one. Similarly, it can see* $\{x|A(x)\}$ *since it can check whether A(x) or not one by one. On the contrary, a human mind is clearly a finite mind.*

"About Mathematics": Several "Minds"

The tasks of the foundations of mathematics for Takeuti;

1. to formulate the function of an infinite mind. (or to find new axioms of set-theory).
2. to justify the world of our infinite mind in the world of a finite mind. We want to confirm of rationality of modern mathematics since the world of our mind is finite and the world of modern mathematics is just conjecture about our infinite mind.
3. to formulate the function of a finite mind. (to develop the formalist standpoint)

Two important points:

- No use of the word "reliable". Use of "rationality" (**「合理性」** in Japanese).
- Set-theory and proof-theory are connected in his thoughts.

Self-Reflection in Finite Mind

Use of Kitaro Nishida's (**西田幾多郎**) another key word "self-reflection" in "About mathematics" in 1972

> *Now, for example, consider a finite mind. If the function of a finite mind is completely finite, then our mathematics remains in finite. However, our finite mind is the so-called potential infinity. In other words, we can indicate infinitary objects like* $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots$ *[...] Why? This is because our mind has the ability of self-reflection, that is, the ability to observe what we are doing and to know what we are doing.*

Self-Reflection in Infinite Mind

Another passage from “About mathematics” in 1972:

> *This holds true for an infinite mind as well. When an infinite mind creates sets from the empty set, though I explain the construction of ordinals very briefly in my lecture or explanation, if we think this over, it is clear that the self-reflection of an infinite mind plays the role here. In particular, the function of self-reflection is most important for a mind which can create something internally when a set is not confined to a closed set and extends further. How to formulate this function of self-reflection of a mind?*

- Similar to Dedekind’s argument.
- Nishida’s early works about it. (Some translations are available)
  - Understanding in logic and understanding in mathematics. **「論理の理解と数理の理解」**, 1912.
- Program to find a new axiom in set-theory is also explained in it.

Takeuti's Finitism Revisited

- New view about Takeuti's finitism based on his math. and phil. papers.
- Working hypothesis: to investigate the relationships between many minds is the task of the found. math.
  - finite, infinite, or constructible minds...
- A key notion mentioned by him: self-reflection.
  - ex. Use of higher-order operations in his well-ordering proof up to ε_0.
- Note: "self-reflection" is a key in Kitaro Nishida's thought.

  Cf. John C. Maraldo, "Nishida Kitaro", *SEP*, https://plato.stanford.edu/entries/nishida-kitaro/

Summary and Questions

- Some key elements of Takeuti's thought explained:
  1. Difference from Hilbert's position. Reliability was not a key for Takeuti.
  2. Use of "mind" and "self-reflection" in Kyoto's school.
- Our working hypothesis: to investigate the relationships between many minds is the task of the found. math.
  - Use of many minds: finite, infinite, or constructible minds...
- Our first paper "TAKEUTI'S PROOF THEORY IN THE CONTEXT OF THE KYOTO SCHOOL" is available at

  `https://www.researchgate.net/publication/332874620_TAKEUTI'S_PROOF_THEORY_IN_THE_CONTEXT_OF_THE_KYOTO_SCHOOL`
- Further work: interpretation of results in proof-theory in this context:
  - to explain the activity of proof-theory to provide a hierarchy of layers of finitism by some complexity in our terms (such as minds, self-reflections...)