First-order theory of lines in Euclidean plane

T. Ivanova[1] T. Tinchev[2]

[1] Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

[2] Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"

Logic Colloquium 2019,
August 11-16, 2019, Prague

This talk is supported by contract DN02/15/19.12.2016 "Space, Time and Modality: Relational, Algebraic and Topological Models" with Bulgarian NSF.

Parallelism and convergence

The paper

P. Balbiani and T. Tinchev, *Line-based affine reasoning in Euclidean plane*, ***Journal of Applied Logic***, vol. 5 (2007), pp. 421–434.

gives qualitative spatial reasoning in Euclidean plane based solely on lines. The relations of parallelism and convergence between lines are considered.

- P - parallel
- C - converge

It is considered the structure

$\langle LE^2; P, C\rangle$,

where LE^2 is the set of all lines in Euclidean plane.

Parallelism and convergence

The axiomatization of the theory of the structure is called SAP and contains the following axioms:

IRREF $\forall x \neg P(x,x) \quad \forall x \neg C(x,x)$
TRANS $\forall x \forall y \forall z (P(x,y) \wedge C(y,z) \rightarrow C(x,z))$
UNIV $\forall x \forall y (x = y \vee P(x,y) \vee C(x,y))$

$DENS_n$

$$\forall x \forall y_1 \ldots \forall y_n (P(x,y_1) \wedge \ldots \wedge P(x,y_n) \rightarrow \exists z (P(x,z) \wedge P(z,y_1) \wedge \ldots \wedge P(z,y_n))), \; n \geq 0$$

$$\forall x \forall y_1 \ldots \forall y_n (C(x,y_1) \wedge \ldots \wedge C(x,y_n) \rightarrow \exists z (C(x,z) \wedge C(z,y_1) \wedge \ldots \wedge C(z,y_n))), \; n \geq 0$$

Parallelism and convergence + perpendicularity

In this talk we consider a continuation of the paper by adding a new predicate - perpendicularity, denoted by O.

We consider the language $\mathcal{L} = \langle\ ;\ ; P, C, O, =\rangle$

and the structure $\langle LE^2; P, C, O\rangle$, where LE^2 is the set of all lines in Euclidean plane

- P - parallel
- C - converge
- O - perpendicular

Outline

- We introduce a first-order theory of lines in Euclidean plane with predicates parallelism, convergence and perpendicularity.
- The logic is complete with respect to the Euclidean plane.
- The theory is ω - categorical and not categorical in every uncountable cardinality.
- We prove that the membership problem of the logic is PSPACE-complete.

Parallelism and convergence + perpendicularity

$$\mu_1 \stackrel{def}{=} \forall x \forall y (O(x,y) \rightarrow O(y,x))$$
$$\mu_2 \stackrel{def}{=} \forall x \forall y \forall z (O(x,z) \wedge O(z,y) \rightarrow (x = y) \vee P(x,y))$$
$$\mu_3 \stackrel{def}{=} \forall x \forall y (O(x,y) \rightarrow C(x,y))$$
$$\mu_4 \stackrel{def}{=} \forall x \exists y O(x,y)$$
$$\mu_5 \stackrel{def}{=} \forall x \forall y \forall z (P(y,x) \wedge O(x,z) \rightarrow O(y,z))$$

We add $\mu_1, \ldots, \mu_5$ to *SAP* and we obtain a theory, called *SAPP*.

Let $\mathcal{A}$ be a model of *SAPP*. $\neg C$ is an equivalence relation in A which splits it into invfinitely many infinite equivalence classes.

We define the relation between equivalence classes $R_{\mathcal{A}}$ in the following way

$[a]R_{\mathcal{A}}[b]$ iff $O(a, b)$ for any $a, b \in A$

For every equivalence class $[a]$ there exists exactly one eqivalence class $[b]$ such that $[a]R_{\mathcal{A}}[b]$.

Proposition

Let $\mathcal{A}$ be a countable model of SAPP, $\mathcal{B}$ be a model of SAPP. Then $\mathcal{A}$ is elementary embeddable in $\mathcal{B}$.

Corollary

SAPP is maximal consistent.

Proposition

SAPP $\vdash \varphi$ iff φ is true in the Euclidean plane.

Proposition

(i) P is not definable with O, C in every model of SAPP;

(ii) O is not definable with P, C in every model of SAPP;

*(iii) The ternary predicate Co ($Co(a, b, c)$ iff there is exactly one point incident with a, b and c)
is not definable in $\langle LE^2; P, C, O\rangle$;*

(iv) C is definable by O in every model of SAPP;

(v) $=$ is definable by P, C in every model of SAPP.

Proposition

SAPP is not finitely axiomatizable.

Proposition

The problem if a closed formula logically follows from SAPP is PSPACE-complete.

1) The problem if a closed formula logically follows from *SAPP* is in *PSPACE*.

$EQ^{\infty} = \{\varphi\colon \varphi$ is closed formula in the language $\mathcal{L}_1$ and φ is true in all infinite structures$\}$, where $\mathcal{L}_1 = \langle ; ; = \rangle$

The membership problem in EQ^{∞} is in *PSPACE*. [Balbiani and Tinchev, 2007]

It is enough to juxtapose to every closed formula φ in $\mathcal{L}$ a closed formula φ_1 in $\mathcal{L}_1$ and to ensure that φ_1 can be obtained from φ algorithmically with use of memory polynomial in the size of φ, and *SAPP* $\models \varphi$ iff $\varphi_1 \in EQ^{\infty}$.

Sketch of the proof

Let $\mathcal{A}^*$ be the substructure of the structure for the language $\mathcal{L}$ with universe the set of all lines in the Euclidean plane, that is obtained by eliminating of the lines, parallel to X-axis, the lines parallel to Y-axis and the lines with equation of the kind $y = bx$.

- $\mathcal{A}^* \models SAPP$
- for any closed formula φ in $\mathcal{L}$: $SAPP \models \varphi$ iff $\mathcal{A}^* \models \varphi$

Sketch of the proof

We consider the language $\mathcal{L}_p$ which is obtained from $\mathcal{L}$ by removing C and $=$.

We juxtapose to every closed formula φ in $\mathcal{L}$ a closed formula φ_p in $\mathcal{L}_p$ by replacing every $C(x, y)$ by $\exists z(O(x, z) \wedge \neg O(y, z))$ and every $x = y$ by $\neg P(x, y) \wedge \forall z(O(x, z) \rightarrow O(y, z))$.

$\mathcal{A}^\star \models \varphi$ iff $\mathcal{A}^\star \models \varphi_p$

Let $\mathcal{R}$ be the structure for $\mathcal{L}_1$ with universe $\mathbb{R} \backslash \{0\}$.

For any closed formula φ in $\mathcal{L}_1$: $\mathcal{R} \models \varphi$ iff $\varphi \in EQ^\infty$

Sketch of the proof

We associate with a line a with equation $y = bx + c$ the numbers $a^1 \stackrel{def}{=} b$, $a^2 \stackrel{def}{=} -1/b$, $a^3 \stackrel{def}{=} c$ ("coordinates").

For φ_p in $\mathcal{L}_p$ we define $\widehat{\varphi_p}$ in $\mathcal{L}_1$ in such way that:
φ_p is true on some lines iff $\widehat{\varphi_p}$ is true on their "coordinates", i.e. for any
$a_1, \ldots, a_n \in A^* \quad \mathcal{A}^* \models \varphi_p[a_1, \ldots, a_n] \Longleftrightarrow \mathcal{R} \models \widehat{\varphi_p}[\overline{a_1}, \ldots, \overline{a_n}]$

Sketch of the proof

- if $\varphi_p = P(x_1, x_2)$, then $\widehat{\varphi_p} \stackrel{def}{=} (x_1^1 = x_2^1) \wedge (x_1^3 \neq x_2^3)$
- if $\varphi_p = O(x_1, x_2)$, then $\widehat{\varphi_p} \stackrel{def}{=} (x_1^1 = x_2^2) \wedge (x_2^1 = x_1^2)$
- if $\varphi_p = \neg\varphi'$, then $\widehat{\varphi_p} \stackrel{def}{=} \neg\widehat{\varphi'}$
- if $\varphi_p = \varphi' \wedge \varphi''$, then $\widehat{\varphi_p} \stackrel{def}{=} \widehat{\varphi'} \wedge \widehat{\varphi''}$

Sketch of the proof

- if $\varphi_p = \exists x_n \varphi'$ and φ' has free variables among $x_1, \ldots, x_n$, then $\widehat{\varphi_p} \stackrel{def}{=} \exists x_n^1 \exists x_n^2 \exists x_n^3 (\widehat{\varphi'} \wedge \kappa_n)$, where for any natural number n κ_n is a formula with free variables $x_1^1, x_1^2, x_2^1, x_2^2, \ldots, x_n^1, x_n^2$, defined in the following way

$$\kappa_n \stackrel{def}{=} \bigwedge_{i<n} \left[(x_i^1 = x_n^1 \leftrightarrow x_i^2 = x_n^2) \wedge (x_i^1 = x_n^2 \leftrightarrow x_n^1 = x_i^2) \right] \wedge (x_n^1 \neq x_n^2)$$

κ_n is true for the coordinates of any line.

Sketch of the proof

By $\overline{a_1}, \ldots, \overline{a_n}$ we denote $a_1^1, a_1^2, a_1^3, \ldots, a_n^1, a_n^2, a_n^3$,
by $\overline{b_1}, \ldots, \overline{b_n}$ we denote $b_1^1, b_1^2, a_1^3, \ldots, b_n^1, b_n^2, a_n^3$.

Lemma

For any formula φ_p in $\mathcal{L}_p$, for any $x_1, \ldots, x_n$ and for any $a_1, \ldots, a_n \in A^\star$, if φ_p has free variables among $x_1, \ldots, x_n$, then

$$\mathcal{A}^* \models \varphi_p[a_1, \ldots, a_n] \text{ iff } \mathcal{R} \models \widehat{\varphi_p}[\overline{a_1}, \ldots, \overline{a_n}]$$

Sketch of the proof

The proof is induction on φ_p.

For $\varphi_p = \exists x_n \varphi'$ and for the direction $\Longleftarrow$ we obtain that there are real numbers, different from 0 - a_n^1, a_n^2, a_n^3 such that $\mathcal{R} \models \widehat{\varphi}' \wedge \kappa_n[\overline{a_1}, \ldots, \overline{a_n}]$

It is possible $a_n^1 \cdot a_n^2 \neq -1$, therefore we cannot apply the induction hypothesis and we must find b_n^1, b_n^2, b_n^3 such that $\mathcal{R} \models \widehat{\varphi}'[\overline{a_1}, \ldots, \overline{a_{n-1}}, \overline{b_n}]$ and $b_n^1 \cdot b_n^2 = -1$.

Sketch of the proof

For this purpose for fixed number triples real numbers, satisfying certain conditions, we will define corresponding to them the same number triples real numbers such that the multiplication of the first two elements of every triple is -1.

We will prove that for any formula φ_p in $\mathcal{L}_p$ $\widehat{\varphi_p}$ is equally true on real numbers and their corresponding (if there are such).

For any natural number n we define a formula ψ_n with free variables $x_1^1, x_1^2, x_2^1, x_2^2, \ldots, x_n^1, x_n^2$ in the following way

$$\psi_n \stackrel{def}{=} \bigwedge_{1 \leq i < j \leq n} \left[(x_i^1 = x_j^1 \leftrightarrow x_i^2 = x_j^2) \wedge (x_i^1 = x_j^2 \leftrightarrow x_j^1 = x_i^2) \right] \wedge \bigwedge_{i=1}^{n} (x_i^1 \neq x_i^2)$$

Sketch of the proof

Definition

Let n be a positive integer. Let $\overline{a_1}, \ldots, \overline{a_n}$ be real numbers, different from 0 such that $\mathcal{R} \models \psi_n[\overline{a_1}, \ldots, \overline{a_n}]$. We will say that the real numbers $\overline{b_1}, \ldots, \overline{b_n}$ are corresponding to $\overline{a_1}, \ldots, \overline{a_n}$, if:

1) for any $i = 1, \ldots, n \quad b_i^1 \cdot b_i^2 = -1$

2) for any $i_1, i_2 \in \{1, \ldots, n\},\ i_1 < i_2$ it is true:

$(a_{i_1}^1 = a_{i_2}^1) \wedge (a_{i_1}^3 \neq a_{i_2}^3)$ iff $(b_{i_1}^1 = b_{i_2}^1) \wedge (b_{i_1}^3 \neq b_{i_2}^3)$
and
$(a_{i_1}^1 = a_{i_2}^2) \wedge (a_{i_2}^1 = a_{i_1}^2)$ iff $(b_{i_1}^1 = b_{i_2}^2) \wedge (b_{i_2}^1 = b_{i_1}^2)$

Sketch of the proof

For the rest of the proof the following two lemmas are enough

Lemma

Let $\overline{b_1}, \ldots, \overline{b_n}$ be corresponding to $\overline{a_1}, \ldots, \overline{a_n}$ and $a^1_{n+1}, a^2_{n+1}, a^3_{n+1}$ be real numbers, different from 0 such that $\mathcal{R} \models \kappa_{n+1}[\overline{a_1}, \ldots, \overline{a_{n+1}}]$. Then there exist real numbers b^1_{n+1}, b^2_{n+1} such that $\overline{b_1}, \ldots, \overline{b_{n+1}}$ are corresponding to $\overline{a_1}, \ldots, \overline{a_{n+1}}$.

Lemma

For any formula φ_p in $\mathcal{L}_p$ we have:

if $\overline{b_1}, \ldots, \overline{b_n}$ are corresponding to $\overline{a_1}, \ldots, \overline{a_n}$, then

$\mathcal{R} \models \widehat{\varphi_p}[\overline{a_1}, \ldots, \overline{a_n}]$ iff $\mathcal{R} \models \widehat{\varphi_p}[\overline{b_1}, \ldots, \overline{b_n}]$.

Sketch of the proof

Lemma

For any formula φ_p in $\mathcal{L}_p$ we have: if φ_p has length n, then $\widehat{\varphi_p}$ has length $\leq 23n^2 - 115n + 377$.

Sketch of the proof

For to prove that the problem if a closed formula logically follows from SAPP is PSPACE-hard, it suffices to prove the following

Lemma

SAPP is a conservative extension of SAP.

Proposition

SAPP is ω-categorical.

Proposition

SAPP is not categorical in any uncountable cardinality.

Future work

- the ternary predicate Co(Co(a,b,c) iff there is exactly one point incident with a,b,c) to be added to the language.
- to be considered higher dimensions

Thank you very much!