The Hajja-Martini inequality in a weak absolute geometry

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Euclid, *Elements*, Proposition I.21

- The first nontrivial geometric inequality (I.21):

  If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

The Hajja-Martini inequality

- (M. Hajja, H. Martini, 2013) Let P be a point in the plane of a triangle ABC. Then there exists a point Q inside or on the boundary of ABC that satisfies

$$AQ \leq AP, \ BQ \leq BP, \ CQ \leq CP.$$

The intuition behind the Hajja-Martini inequality

- Imagine a rigid (say, wooden) triangle ABC, held by a needle positioned in a point P on the plane of the triangle, holding three threads connecting it to the three vertices A, B, C (where we can imagine needle ears being attached). If P lies outside the triangle, then we can imagine lifting the needle from the plane of ABC and ending up with a 3-dimensional situation, a tetrahedron with a wooden base and thread edges connecting the needle in P (now outside the plane of ABC) to the vertices A, B, C. It is plain that the thread can be pulled from A, B, C, thus shortening PA, PB, PC, to bring P down to the plane determined by ABC (even inside ABC). In case P lies inside ABC, no such shortening is possible.

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- Hajja and Martini prove it inside plane Euclidean geometry over the reals using Zorn's Lemma and Bolzano-Weierstrass.
- And wonder whether that machinery is really needed.

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- Such concerns, over the *purity of the method* of proof go back, in modern times, to H. Wiener (1890):
- *One can ask of the proof of a mathematical theorem that it uses only those assumptions on which the theorem really depends. The least imaginable assumptions are the existence of certain objects and certain operations by which those objects are connected. If it is possible to string together such objects and operations, without adding new assumptions, in such a manner that theorems arise, then one obtains in these theorems a* self-contained domain *of science.*

David Hilbert

- Hilbert (1899), the last page of his *Foundations of Geometry*:

  The tenet, according to which one should clarify the principles of the possibility of proofs, is intimately connected with the requirement of the "purity" of the methods of proof, which have been emphasized of late by several mathematicians. This requirement is, after all, nothing else than the subjective form of the tenet followed here. In effect, the analysis performed here searches in general to shed light on the question regarding which axioms, hypotheses or auxiliary means are necessary for the proof of an elementary geometric truth

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- for every pair of points (A, B), there is a point $\mu_0(A, B)$ which lies between A and B and is referred to as the *midpoint* of AB $(\mu_0(A, B) = \mu_0(B, A)$ for all $A \neq B)$

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- $AB < AC$ is defined by asking that $B \neq C$ and that the perpendicular raised in $\mu_0(B, C)$ to the line BC shuld intersect the open segment AC. We ask that $<$ be transitive, i.e., that $AB < AC \wedge AC < AD \rightarrow AB < AD$

Enough to prove the Hajja-Martini inequality

- For every point P outside of triangle ABC there exists a point Q inside or on the boundary of triangle ABC, such that Q and P satisfy

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- If P lies inside ABC, then there is no $Q \neq P$ such that

$$AQ \leq AP, \; BQ \leq BP, \; CQ \leq CP.$$

  holds

Weak absolute geometry

- Although it is “apparent” that our axiom system is rudimentary, much weaker than Hilbert’s axiom system for absolute geometry, we could not find a model satisfying all these axioms, that would not be a model of absolute geometry.