Cohesive powers of $\omega$

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Cohesive sets

Let
\[ \vec{A} = (A_0, A_1, A_2, \ldots) \]
be a countable sequence of subsets of \( \mathbb{N} \).

Then there is an \textbf{infinite} set \( C \subseteq \mathbb{N} \) such that, for every \( i \):

- either \( C \subseteq^* A_i \)
- or \( C \subseteq^* \overline{A}_i \).

\( C \) is called \textbf{cohesive} for \( \vec{A} \), or simply \( \vec{A} \)-cohesive.

If \( \vec{A} \) is the sequence of recursive sets, then \( C \) is called \textbf{r-cohesive}.

If \( \vec{A} \) is the sequence of r.e. sets, then \( C \) is called \textbf{cohesive}.
Skolem’s countable non-standard model of true arithmetic

Skolem (1934):
Let $C$ be cohesive for the sequence of arithmetical sets.
(Such a $C$ is also called \textit{arithmetically indecomposable}.)

Consider arithmetical functions $f, g : \mathbb{N} \to \mathbb{N}$. Define:

$f =_C g$ if $C \subseteq^* \{ n : f(n) = g(n) \}$

$f < g$ if $C \subseteq^* \{ n : f(n) < g(n) \}$

$(f + g)(n) = f(n) + g(n)$

$(f \times g)(n) = f(n) \times g(n)$

Let $[f] = \{ g : g =_C f \}$ denote the $=_C$-equivalence class of $f$.

Form a structure $\mathcal{M}$ with domain $\{ [f] : f \text{ arithmetical} \}$ and

$[f] < [g]$ if $f < g$; $[f] + [g] = [f + g]$; $[f] \times [g] = [f \times g]$.

Then $\mathcal{M}$ models true arithmetic!
**Effectivizing Skolem’s construction**

**Tennenbaum wanted to know:**
What if we did Skolem’s construction, but

- used recursive functions $f: \mathbb{N} \rightarrow \mathbb{N}$ in place of arithmetical functions;
- only assumed that $C$ is r-cohesive?

Do we still get models of true arithmetic?

**Feferman-Scott-Tennenbaum (1959):**
It is not even possible to get models of Peano arithmetic in this way.

**Lerman (1970)** has further results in this direction:
If you only consider **co-maximal** sets $C$, then the structure you get depends only on the many-one degree of $C$.

**(Co-maximal** means co-r.e. and cohesive.)
Dimitrov (2009):
Let $\mathcal{A}$ be a computable structure.
(i.e., $\mathcal{A}$ has domain $\mathbb{N}$ and recursive functions and relations.)

Let $C$ be cohesive. Form the cohesive power $\Pi_C \mathcal{A}$ of $\mathcal{A}$ by $C$:

Consider partial recursive $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ with $C \subseteq^* \text{dom}(\varphi)$. Define:

$$\varphi =_C \psi \quad \text{if} \quad C \subseteq^* \{ n : \varphi(n) = \psi(n) \}$$
$$R(\psi_0, \ldots, \psi_{k-1}) \quad \text{if} \quad C \subseteq^* \{ n : R(\psi_0(n), \ldots, \psi_{k-1}(n)) \}$$
$$F(\psi_0, \ldots, \psi_{k-1})(n) = F(\psi_0(n), \ldots, \psi_{k-1}(n))$$

Let $[\varphi]$ denote the $=_C$-equivalence class of $\varphi$.

Let $\Pi_C \mathcal{A}$ be the structure with domain $\{ [\varphi] : C \subseteq^* \text{dom}(\varphi) \}$ and

$$R([\psi_0], \ldots, [\psi_{k-1}]) \quad \text{if} \quad R(\psi_0, \ldots, \psi_{k-1})$$
$$F([\psi_0], \ldots, [\psi_{k-1}]) = [F(\psi_0, \ldots, \psi_{k-1})].$$
A little Łoś

For cohesive powers:

1. Łoś’s theorem holds for $\Sigma_2$ sentences and $\Pi_2$ sentences.
2. A one-way Łoś’s theorem holds for $\Pi_3$ sentences.

**Theorem (Łoś’s theorem for cohesive powers; Dimitrov)**

Let $\mathcal{A}$ be a computable structure, and let $C$ be cohesive. Then

1. If $\theta$ is a $\Sigma_2$ sentence or a $\Pi_2$ sentence, then
   
   $$\Pi_C \mathcal{A} \models \theta \iff \mathcal{A} \models \theta$$

2. If $\theta$ is a $\Pi_3$ sentence, then
   
   $$\Pi_C \mathcal{A} \models \theta \implies \mathcal{A} \models \theta$$
A Quirky observation

Consider $\mathbb{Q}$ as a linear order (i.e., as a structure in the language \{$\lt$\}).

$\mathbb{Q}$ is a countable dense linear order without endpoints.

If $\mathcal{L}$ is a countable dense linear order without endpoints, then $\mathcal{L} \cong \mathbb{Q}$.

“Dense linear order w/o endpoints” is axiomatized by a $\Pi_2$ sentence $\theta$.

If $C$ is any cohesive set, then $\Pi_C \mathbb{Q} \models \theta$ by Łoś for cohesive powers.

So $\Pi_C \mathbb{Q}$ is a countable dense linear order without endpoints.

Thus $\Pi_C \mathbb{Q} \cong \mathbb{Q}$.

So it is possible for every cohesive power of $\mathbb{A}$ to be isomorphic to $\mathbb{A}$!

(Not an accident: $\Pi_C \mathbb{A}$ will be isomorphic to $\mathbb{A}$ whenever $\mathbb{A}$ is ultrahomogeneous in a sufficiently effective way.)
What about cohesive powers of $\mathbb{N}$?

Terminology:
- Still considering linear orders (i.e., the language $\{<\}$).
- Let ‘$\mathbb{N}$’ denote the usual presentation of $\mathbb{N}$.
- Say that $\mathcal{L}$ is a **recursive copy** of $\mathbb{N}$ if $\mathcal{L}$ is a recursive linear order and $\mathcal{L} \cong \mathbb{N}$ (possibly by a **non-recursive** isomorphism).

Can check:
- If $C$ is cohesive, then $\Pi_C \mathbb{N} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.
- If $C$ is cohesive and $\mathcal{L} \cong \mathbb{N}$ via a recursive isomorphism, then $\Pi_C \mathcal{L} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.

(Recall that $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ is the order-type of countable non-standard models of PA.)

(Here, $\mathbb{Q} \times \mathbb{Z}$ denotes the lexicographic order on $\mathbb{Q} \times \mathbb{Z}$. I think $\mathbb{Q} \times \mathbb{Z}$ is easier to read than $\mathbb{Z} \mathbb{Q}$.)
Are there other cohesive powers of $\mathbb{N}$?

More properly:
Is there a recursive copy $\mathcal{L}$ of $\mathbb{N}$ with $\Pi_C \mathcal{L} \not\cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$?

Such an $\mathcal{L}$ cannot be isomorphic to $\mathbb{N}$ via a recursive isomorphism.

Classic recursive copy $\mathcal{L} = (\mathbb{N}, \prec)$ of $\mathbb{N}$ with non-recursive isomorphism:

• Let $f : \mathbb{N} \to \mathbb{N}$ be recursive injection with $\text{ran}(f) = K = \{ e : \Phi_e(e) \downarrow \}$.

• Put the evens in their usual order: $2a \prec 2b$ if $2a < 2b$.

• For each $s$, put $2s + 1$ between $2f(s)$ and $2f(s) + 2$:
  $2f(s) \prec 2s + 1 \prec 2f(s) + 2$.

However:
With this example, we still get $\Pi_C \mathcal{L} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ for every cohesive $C$.

So it is not enough just to ensure that the isomorphism $\mathcal{L} \cong \mathbb{N}$ is non-recursive!
A different cohesive power of $\mathbb{N}$

**Theorem (D H M So Sh V)**

For every co-r.e. cohesive set $C$, there is a recursive copy $\mathcal{L}$ of $\mathbb{N}$ such that

$$\Pi_C \mathcal{L} \not\simeq \mathbb{N} + (\mathbb{Q} \times \mathbb{Z}).$$

**Idea:**
Build $\mathcal{L} = (\mathbb{N}, \prec)$ so that $[\text{id}]$ does not have an immediate successor in the cohesive power $\Pi_C \mathcal{L}$.

To do this, ensure that $\varphi_e(n)$ is not the $\prec$-immediate successor of $n$ for almost every $n \in C$:

$$\forall^\infty n \in C \ (\varphi_e(n) \downarrow \Rightarrow \varphi_e(n) \text{ is not the } \prec\text{-immediate successor of } n)$$

Then $[\varphi_e]$ is not the immediate successor of $[\text{id}]$ in $\Pi_C \mathcal{L}$. 
Hints of the construction

$C$ is co-r.e., so fix an infinite recursive $R \subseteq \overline{C}$.

The elements of $R$ are safe:

It does not matter if $\varphi_e(n)$ is the $\prec$-immediate successor of $n$ for $n \in R$ because these $n$ are not in $C$.

Define $\mathcal{L} = (\mathbb{N}, \prec)$ in stages.

At each stage, $\prec$ will have been defined on a finite set.

At stage $s$:

- If $\prec$ is not yet defined on $s$, make $s$ $\prec$-greatest of what we have so far.
- Examine the pairs $\langle e, n \rangle < s$. If
  - $n \notin R$,
  - $\varphi_{e,s}(n) \downarrow$,
  - $\varphi_e(n)$ is currently the $\prec$-immediate successor of $n$, and
  - $n$ is not $\prec$-below any of $0, 1, \ldots, e$

  then choose a fresh $m$ from $R$ and define $n \prec m \prec \varphi_e(n)$. 

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We can enhance the construction to make the non-standard elements of $\Pi_C \mathcal{L}$ be $\mathbb{Q}$.

**Theorem** (D H M So Sh V) 
For every co-r.e. cohesive set $C$, there is a recursive copy $\mathcal{L}$ of $\mathbb{N}$ such that 

$$\Pi_C \mathcal{L} \cong \mathbb{N} + \mathbb{Q}.$$ 

(Theorem* is still being checked by some of the co-authors!)

This theorem leads to **even more examples**! 
(Again given a co-r.e. cohesive $C$.)

- $\mathcal{L} \times 2 \cong \mathbb{N}$ and $\Pi_C (\mathcal{L} \times 2) \cong \mathbb{N} + (\mathbb{Q} \times 2)$
- For any finite sequence of finite linear orders $L_0, \ldots, L_n$, there is a recursive copy $\mathcal{J}$ of $\mathbb{N}$ with 

$$\Pi_C \mathcal{J} \cong \mathbb{N} + \text{the shuffle of } L_0, \ldots, L_n.$$
Thank you for coming to my talk! Do you have a question about it?