Cohesive powers of ω

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Cohesive sets

Let

$$\vec{A} = (A_0, A_1, A_2, \dots)$$

be a countable sequence of subsets of $\mathbb{N}$.

Then there is an **infinite** set $C \subseteq \mathbb{N}$ such that, for every i:

$$\text{either } C \subseteq^* A_i$$
$$\text{or } C \subseteq^* \overline{A_i}.$$

C is called **cohesive** for $\vec{A}$, or simply $\vec{A}$-**cohesive**.

If $\vec{A}$ is the sequence of recursive sets, then C is called **r-cohesive**.

If $\vec{A}$ is the sequence of r.e. sets, then C is called **cohesive**.

Skolem's countable non-standard model of true arithmetic

Skolem (1934):
Let C be cohesive for the sequence of arithmetical sets.
(Such a C is also called **arithmetically indecomposable**.)

Consider arithmetical functions $f, g\colon \mathbb{N} \to \mathbb{N}$. Define:

$$\begin{array}{lcl} f =_C g & \text{if} & C \subseteq^* \{n : f(n) = g(n)\} \\ f < g & \text{if} & C \subseteq^* \{n : f(n) < g(n)\} \\ (f+g)(n) & = & f(n) + g(n) \\ (f \times g)(n) & = & f(n) \times g(n) \end{array}$$

Let $[f] = \{g : g =_C f\}$ denote the $=_C$-equivalence class of f.

Form a structure $\mathfrak{M}$ with domain $\{[f] : f \text{ arithmetical}\}$ and

$$[f] < [g] \text{ if } f < g; \qquad [f] + [g] = [f+g]; \qquad [f] \times [g] = [f \times g].$$

Then $\mathfrak{M}$ models true arithmetic!

Effectivizing Skolem's construction

Tennenbaum wanted to know:
What if we did Skolem's construction, but

- used recursive functions $f\colon \mathbb{N} \to \mathbb{N}$ in place of arithmetical functions;
- only assumed that C is r-cohesive?

Do we still get models of true arithmetic?

Feferman-Scott-Tennenbaum (1959):
It is not even possible to get models of Peano arithmetic in this way.

Lerman (1970) has further results in this direction:

If you only consider **co-maximal** sets C, then the structure you get depends only on the many-one degree of C.

(**Co-maximal** means co-r.e. and cohesive.)

Cohesive powers

Dimitrov (2009):

Let $\mathfrak{A}$ be a computable structure.
(i.e., $\mathfrak{A}$ has domain $\mathbb{N}$ and recursive functions and relations.)

Let C be cohesive. Form the **cohesive power** $\Pi_C \mathfrak{A}$ of $\mathfrak{A}$ by C:

Consider partial recursive $\varphi, \psi \colon \mathbb{N} \to \mathbb{N}$ with $C \subseteq^* \mathrm{dom}(\varphi)$. Define:

$$\begin{array}{lcl} \varphi =_C \psi & \text{if} & C \subseteq^* \{n : \varphi(n) = \psi(n)\} \\ R(\psi_0, \dots, \psi_{k-1}) & \text{if} & C \subseteq^* \{n : R(\psi_0(n), \dots, \psi_{k-1}(n))\} \\ F(\psi_0, \dots, \psi_{k-1})(n) & = & F(\psi_0(n), \dots, \psi_{k-1}(n)) \end{array}$$

Let $[\varphi]$ denote the $=_C$-equivalence class of φ.

Let $\Pi_C \mathfrak{A}$ be the structure with domain $\{[\varphi] : C \subseteq^* \mathrm{dom}(\varphi)\}$ and

$$\begin{array}{lcl} R([\psi_0], \dots, [\psi_{k-1}]) & \text{if} & R(\psi_0, \dots, \psi_{k-1}) \\ F([\psi_0], \dots, [\psi_{k-1}]) & = & [F(\psi_0, \dots, \psi_{k-1})]. \end{array}$$

A little Łoś

For cohesive powers:

1. Łoś's theorem holds for Σ_2 sentences and Π_2 sentences.
2. A one-way Łoś's theorem holds for Π_3 sentences.

Theorem (Łoś's theorem for cohesive powers; Dimitrov)

Let $\mathfrak{A}$ be a computable structure, and let C be cohesive. Then

1. *If θ is a Σ_2 sentence or a Π_2 sentence, then*

$$\Pi_C\mathfrak{A} \models \theta \quad \textit{if and only if} \quad \mathfrak{A} \models \theta$$

2. *If θ is a Π_3 sentence, then*

$$\Pi_C\mathfrak{A} \models \theta \quad \textit{implies} \quad \mathfrak{A} \models \theta$$

A $\mathbb{Q}$uirky observation

Consider $\mathbb{Q}$ as a linear order (i.e., as a structure in the language $\{<\}$.)

$\mathbb{Q}$ is a countable dense linear order without endpoints.

If $\mathfrak{L}$ is a countable dense linear order without endpoints, then $\mathfrak{L} \cong \mathbb{Q}$.

"Dense linear order w/o endpoints" is axiomatized by a Π_2 sentence θ.

If C is any cohesive set, then $\Pi_C \mathbb{Q} \models \theta$ by Łoś for cohesive powers.

So $\Pi_C \mathbb{Q}$ is a countable dense linear order without endpoints.

Thus $\Pi_C \mathbb{Q} \cong \mathbb{Q}$.

So it is possible for every cohesive power of $\mathfrak{A}$ to be isomorphic to $\mathfrak{A}$!

(Not an accident: $\Pi_C \mathfrak{A}$ will be isomorphic to $\mathfrak{A}$ whenever $\mathfrak{A}$ is ultrahomogeneous in a sufficiently effective way.)

What about cohesive powers of $\mathbb{N}$?

Terminology:

- Still considering linear orders (i.e., the language $\{<\}$).
- Let '$\mathbb{N}$' denote the usual presentation of $\mathbb{N}$.
- Say that $\mathfrak{L}$ is a **recursive copy** of $\mathbb{N}$ if $\mathfrak{L}$ is a recursive linear order and $\mathfrak{L} \cong \mathbb{N}$ (possibly by a **non-recursive** isomorphism).

Can check:

- If C is cohesive, then $\Pi_C \mathbb{N} \;\cong\; \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.
- If C is cohesive and $\mathfrak{L} \cong \mathbb{N}$ via a recursive isomorphism, then $\Pi_C \mathfrak{L} \;\cong\; \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.

(Recall that $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ is the order-type of countable non-standard models of PA.)

(Here, $\mathbb{Q} \times \mathbb{Z}$ denotes the lexicographic order on $\mathbb{Q} \times \mathbb{Z}$. I think $\mathbb{Q} \times \mathbb{Z}$ is easier to read than $\mathbb{Z}\mathbb{Q}$.)

Are there other cohesive powers of $\mathbb{N}$?

More properly:
Is there a recursive copy $\mathfrak{L}$ of $\mathbb{N}$ with $\Pi_C \mathfrak{L} \not\cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$?

Such an $\mathfrak{L}$ cannot be isomorphic to $\mathbb{N}$ via a recursive isomorphism.

Classic recursive copy $\mathfrak{L} = (\mathbb{N}, \prec)$ of $\mathbb{N}$ with non-recursive isomorphism:

- Let $f \colon \mathbb{N} \to \mathbb{N}$ be recursive injection with $\mathrm{ran}(f) = K = \{e : \Phi_e(e)\downarrow\}$.
- Put the evens in their usual order: $2a \prec 2b$ if $2a < 2b$.
- For each s, put $2s+1$ between $2f(s)$ and $2f(s)+2$:
  $2f(s) \prec 2s+1 \prec 2f(s)+2$.

However:
With this example, we still get $\Pi_C \mathfrak{L} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ for every cohesive C.

So it is **not enough** just to ensure that the isomorphism $\mathfrak{L} \cong \mathbb{N}$ is non-recursive!

A different cohesive power of $\mathbb{N}$

Theorem (D H M So Sh V)

For every co-r.e. cohesive set C, there is a recursive copy $\mathfrak{L}$ of $\mathbb{N}$ such that

$$\Pi_C \mathfrak{L} \ \not\cong \ \mathbb{N} + (\mathbb{Q} \times \mathbb{Z}).$$

Idea:
Build $\mathfrak{L} = (\mathbb{N}, \prec)$ so that $[\mathrm{id}]$ does **not** have an immediate successor in the cohesive power $\Pi_C \mathfrak{L}$.

To do this, ensure that $\varphi_e(n)$ is **not** the $\prec$-immediate successor of n for almost every $n \in C$:

$$\forall^\infty n \in C \ (\varphi_e(n)\downarrow \Rightarrow \varphi_e(n) \text{ is **not** the } \prec\text{-immeidate successor of } n)$$

Then $[\varphi_e]$ is **not** the immediate successor of $[\mathrm{id}]$ in $\Pi_C \mathfrak{L}$.

Hints of the construction

C is co-r.e., so fix an infinite recursive $R \subseteq \overline{C}$.

The elements of R are **safe**:

It **does not matter** if $\varphi_e(n)$ is the $\prec$-immediate successor of n for $n \in R$ because these n are **not** in C.

Define $\mathfrak{L} = (\mathbb{N}, \prec)$ in stages.

At each stage, $\prec$ will have been defined on a finite set.

At stage s:

- If $\prec$ is not yet defined on s, make s $\prec$-greatest of what we have so far.
- Examine the pairs $\langle e, n \rangle < s$. If
  - $n \notin R$,
  - $\varphi_{e,s}(n)\downarrow$,
  - $\varphi_e(n)$ is currently the $\prec$-immediate successor of n, and
  - n is not $\prec$-below any of $0, 1, \dots, e$

  then choose a fresh m from R and define $n \prec m \prec \varphi_e(n)$.

Making a mess of the non-standards

We can enhance the construction to make the non-standard elements of $\Pi_C \mathfrak{L}$ be $\mathbb{Q}$.

Theorem* (D H M So Sh V)

For every co-r.e. cohesive set C, there is a recursive copy $\mathfrak{L}$ of $\mathbb{N}$ such that

$$\Pi_C \mathfrak{L} \ \cong \ \mathbb{N} + \mathbb{Q}.$$

(Theorem* is still being checked by some of the co-authors!)

This theorem leads to **even more examples**!

(Again given a co-r.e. cohesive C.)

- $\mathfrak{L} \times 2 \ \cong \ \mathbb{N}$ and $\Pi_C(\mathfrak{L} \times 2) \ \cong \ \mathbb{N} + (\mathbb{Q} \times 2)$
- For any finite sequence of finite linear orders $L_0, \ldots, L_n$, there is a recursive copy $\mathfrak{J}$ of $\mathbb{N}$ with

$$\Pi_C \mathfrak{J} \ \cong \ \mathbb{N} + \text{the shuffle of } L_0, \ldots, L_n.$$

Thank you!

Thank you for coming to my talk!
Do you have a question about it?