Overview of Feedback, and the Feedback Hyperjump

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Feedback Turing Computability

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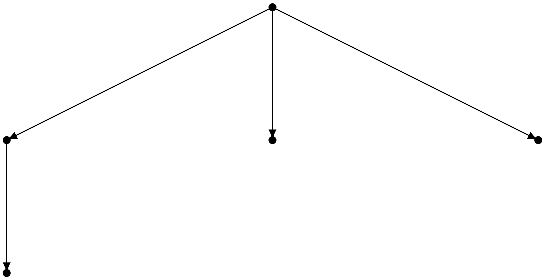
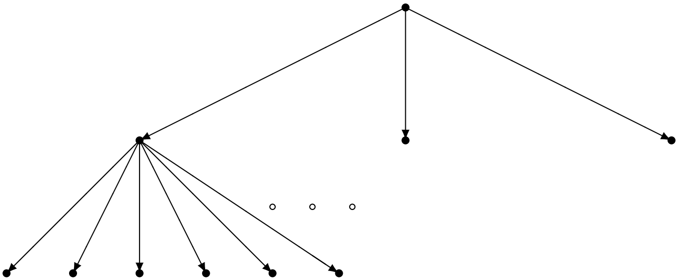
Any fixed point of the operator $X \mapsto H_X$ gives a coherent notion of feedback. The easiest semantics is the *least fixed point*.

Tree of Sub-Computations



Another Good Example



A Bad Example

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One can naturally define the course of a computation if and only if the tree of sub-computations is well-founded.

Feedback Turing Machines

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So some legal computations have an undefined result: the **freezing** computations. The **non-freezing** computations have a perfectly well-defined semantics.
Notation: $\langle e \rangle(n)$

Other Kinds of Feedback

- Feedback Primitive Recursion: Let h be the smallest function such that $h(e, \vec{n}) = \{e\}^h(\vec{n})$, e a code for a primitive recursive function.

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- Feedback Turing on Cantor Space: Let $f(Y) : \mathcal{C} \to \mathcal{C}$ be $\langle e \rangle^Y$.
- Parallel Feedback Turing Computability: Allows oracle questions of the form $\{e\}(\cdot)$?, with answer some $\{e\}(n) = k$.

Theorems

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(AFL) Y is feedback Turing computable iff Y is hyperarithmetic iff Y is Δ^1_1 iff $Y \in L_{\omega_1^{CK}}$.

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(L) Y is parallel feedback Turing computable iff $Y \in L_\gamma$, where γ is the least ordinal which is Π_1 gap-reflection on admissibles.

Gap-Reflection

Definition

Given δ, consider $\phi(\delta)$ a Π_1 sentence with parameters δ and members of L_δ. Then δ is Π_1 **gap-reflecting on admissibles** if for all such ϕ, if $L_{\delta^+} \models \phi(\delta)$, then for some $\beta < \delta$ $L_{\beta^+} \models \phi(\beta)$.

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The least such ordinal, γ, is

- the least Σ^1_1-reflecting ordinal,
- the closure of Σ_2 definable sets in the μ-calculus,
- the closure of Σ^1_1 monotone inductive definitions,
- the least ordinal over which Σ^0_2 Determinacy holds, and
- the least ordinal with the Σ^1_1 Ramsey property.

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$n \in \mathcal{O}$ iff T_n is (well-formed and) well-founded.

Strict $\mathcal{O}$

Relativize: Let $H_X(n) = \downarrow$ resp. $\uparrow$ iff T_n^X is well- resp. ill-founded, where T_n^X must be fully defined (i.e. not freezing).
Let $\mathcal{SO}$ be the least fixed point.

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Let $\mathcal{SO}$ be the least fixed point.
Conjecture/Theorem: A set is computable from $\mathcal{SO}$ iff it is in L_α, where α is the least recursively inaccessible.

Loose $\mathcal{O}$

Let $H_X(n) = \downarrow$ iff T_n^X is well-founded, where T_n^X must be non-freezing, and
$H_X(n) = \uparrow$ iff T_n^X is ill-founded (even if T_n^X is freezing, a kind of tree parallelism).
Let $\mathcal{LO}$ be the least fixed point.

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Conjecture/Theorem: A set is computable from $\mathcal{LO}$ iff it is in L_γ, where γ is the least ordinal which is Π_1 gap-reflecting on admissibles.

Feedback for Other Notions of Computability

Least fixed point semantics for other kinds of computability, such as:

- K_2 computability,
- E-recursion,
- Lifschitz computability,
- infinitary and register machines,
- graph models.

Other Fixed Points

Example

Feedback Turing: Recall the monotone inductive operator $H_X(e) = \uparrow$ resp. $\downarrow$ iff $\{e\}^X \uparrow$ resp. $\downarrow$.
Take the least fixed point. Set all freezing computations to "divergent" and iterate H_X to a fixed point. Repeat, until you have a fixed point of that operation. What does that compute?

Iterated Feedback

Example

Let a second oracle tell you when computations relative to the first oracle are freezing (level 0 and level 1 freezing). What does that compute?

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Iterate levels of freezing along any ordinal. What does that compute?

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Iterate levels of freezing along a ordinal built dynamically during the computation. What does that compute?

Feedback along an Order

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Extend the definition of iteration along an ordinal to iteration along any partial order. For interesting partial orders (e.g. the rationals), what does that compute? What does this compute along any partial order built dynamically?

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Example

Extend the definition of iteration along a partial order to iteration along an order. For instance, two oracles, each of which gives freezing information about the other. What does that compute? What kind of information does that yield?

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