Big Ramsey Degrees and Equivalence Relations

John Howe

University of Leeds

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Outline

Introduction

Theorem (Ramsey '30)

For any $n, k \in \omega$, $\omega \to (\omega)^n_{k,1}$. *That is, for any colouring* $c : [\omega]^n \to k$ *there is an infinite subset* $X \subseteq \omega$ *for which* $|c([X]^n)| \leq 1$.

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Motivating Question: What happens if we consider some countable homogeneous structure instead of ω?

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There is a colouring $c : [\mathbb{Q}]^2 \to 2$ *which takes both colours on any* $X \subseteq \mathbb{Q}$ *with* $(X, <) \cong (\mathbb{Q}, <)$.

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Theorem (Galvin '68, Devlin '79)

For the rationals as a linear order and for each finite k, $\mathbb{Q} \to (\mathbb{Q})^2_{k,2}$. *That is, for any colouring* $c : [\mathbb{Q}]^2 \to k$, *there is* $X \subseteq \mathbb{Q}$ *with* $(X, <) \cong (\mathbb{Q}, <)$ *and* $|c([X]^2)| \leq 2$.

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In our terms, the combination of the previous two results says that the Big Ramsey Degree of the 2 element linear order in $\mathbb{Q}$ is 2.

Definition

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For a countable structure $\mathbb{A}$ *and a substructure* B *of* $\mathbb{A}$*, denote by* $\binom{\mathbb{A}}{B}$ *the collection of induced substructures of* $\mathbb{A}$ *isomorphic to* B*.*
The partition relation $\mathbb{A} \to (\mathbb{A})^{B}_{k,\ell}$ *means that for any colouring* $c : \binom{\mathbb{A}}{B} \to k$*, there is* $\mathbb{A}' \in \binom{\mathbb{A}}{\mathbb{A}}$ *such that* c *takes at most* ℓ *colours on* $\binom{\mathbb{A}'}{B}$

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Definition (Kechris, Pestov, Todorčević '05)

Let $\mathbb{A}$ be a countable structure in some first order language $\mathcal{L}$. For B a finite substructure of $\mathbb{A}$, if there is some finite ℓ such that $\mathbb{A} \to (\mathbb{A})^B_{k,\ell}$ holds for each $k \in \omega$, we say the Big Ramsey Degree (BRD) of B in $\mathbb{A}$ is the least such ℓ. We write this as $T_{\mathbb{A}}(B) = \ell$.

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We say that $\mathbb{A}$ *has finite Big Ramsey Degrees if* $T_{\mathbb{A}}(B)$ *exists for each finite substructure* B *of* $\mathbb{A}$*.*

Existing Results

The following structures have been shown to have finite Big Ramsey Degrees:

- Pure Set $\forall n T(n) = 1$ (Ramsey '30).

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There are also some connections to Topological Dynamics as shown by (Zucker, '17).

Generic Convex Equivalence Relation

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Theorem (H.)

$\mathbb{Q}^c_\sim$ *has finite Big Ramsey Degrees.*

This is proved by applying the existence of BRDs for $\mathbb{Q}$ in the original proof for the generic equivalence relation.

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$\mathbb{Q}_\sim$ *has finite Big Ramsey Degrees.*

In order to show this we combine the techniques for the rationals and equivalence relations.

The Rationals

The lexicographic order on $2^{<\omega}$ is isomorphic to $(\mathbb{Q}, <)$. So the following Ramsey theorem about trees enables us to determine the Big Ramsey Degrees of $(\mathbb{Q}, <)$.

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Definition

For $N \leq \omega$, a Strong Subtree of height N of $2^{<\omega}$ is a meet-closed subset S of $2^{<\omega}$ such that there are lengths $l_n (n < N)$ satisfying:

1. *Each $s \in S$ has length l_n for some $n < N$.*
2. *If $n + 1 < N$ and s has length l_n then $s^\frown 0$ and $s^\frown 1$ have unique extensions to l_{n+1}.*

Let $\mathcal{S}_N(2^{<\omega})$ denote the collection of strong subtrees of height N.

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Theorem (Milliken '79)

For any $n < \omega$ and for any colouring $\mathcal{S}_n(2^{<\omega}) \to k$ there is $S \in \mathcal{S}_\omega(2^{<\omega})$ on which the colouring is monochromatic.

The Generic Equivalence Relation

This is the countable equivalence relation with infinitely many infinite classes.

Definition

Let $\mathbb{E} = (\omega, \prec, \sim, f)$ *be a structure where*

1. $\prec$ *is the usual ordering on* ω*,*
2. $\sim$ *is a generic equivalence relation, and*
3. f *is a function taking* x *to the* $\prec$*-first point in its class.*

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For E *a finite substructure of* $\mathbb{E}$ *and a colouring* $c : \binom{\mathbb{E}}{E} \to k$*, there is* $\mathbb{D} \in \binom{\mathbb{E}}{\mathbb{E}}$ *on which the colouring is monochromatic.*

If, for nodes $s, t \in 2^{<\omega}$, we define the lexicographic order $<$ and let $s \sim t$ if $|s| \sim |t|$ in $\mathbb{E}$, then the structure $(2^{<\omega}, <, \sim)$ is isomorphic to $\mathbb{Q}_{\sim}$.

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The following Theorem plays the role that Milliken's Theorem did in $\mathbb{Q}$ in determining the Ramsey Degrees for $\mathbb{Q}_\sim$.

Theorem (H.)

For any finite substructure E of $\mathbb{E}$ and for any colouring $\mathcal{S}_E(2^{<\omega}) \to k$ there is $S \in \mathcal{S}_{\mathbb{E}}(2^{<\omega})$ on which the colouring is monochromatic.

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Thank You!