Computable Reducibility, and Isomorphisms of Distributive Lattices

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Computable reducibility

In the talk, we consider equivalence relations on the domain $\mathbb{N}$.

Let E and F be equivalence relations. The relation E is **computably reducible** to F, denoted by $E \leq_c F$, if there is a total computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that for any $m, n \in \mathbb{N}$,

$$(m\,E\,n) \;\Leftrightarrow\; (f(m)\,F\,f(n)).$$

The systematic study of computable reducibility was initiated by Ershov (1971). His approach stems from the theory of numberings.

(i) On one hand, computable reducibility is a *stronger* version of m-reducibility.

Let $A, B \subseteq \mathbb{N}$. Recall that $A \leq_m B$ if there is a total computable function $f(x)$ such that for all $x \in \mathbb{N}$,

$$x \in A \;\Leftrightarrow\; f(x) \in B.$$

(ii) On the other hand, one can treat computable reducibility as an effective version of Borel reducibility.

Let X and Y be Polish spaces. Suppose that R and S are equivalence relations on X and Y, respectively. Then R is Borel reducible to S if there is a Borel map $f\colon X \to Y$ such that for all $x, y \in X$,

$$(x\, R\, y) \;\Leftrightarrow\; (f(x)\, S\, f(y)).$$

Computable reducibility for ceers

An equivalence relation E is a *ceer* if E is computably enumerable.

By Id we denote the identity relation on $\mathbb{N}$. For a number $n \geq 1$,

$$\mathrm{Id}_n = \{(x, y) : n \text{ divides } (x - y)\}.$$

- Any ceer with precisely n equivalence classes is c-equivalent to Id_n.
- Every Id_n is c-reducible to any ceer with infinitely many classes.
- There exists a *universal ceer*, i.e. a ceer U such that $E \leq_c U$ for any ceer E.
- The poset of the c-degrees of ceers is neither an upper semilattice, nor a lower semilattice [Andrews, Lempp, J. Miller, Ng, San Mauro, and Sorbi 2014].

For a computable signature L, one can identify first-order L-formulas with their Gödel numbers.

Let T be a first-order L-theory. Then the *provable equivalence in* T is defined as follows. For L-sentences ϕ and ψ,

$$\phi \sim_T \psi \;\Leftrightarrow\; T \vdash (\phi \leftrightarrow \psi).$$

Theorem (Bernardi and Sorbi 1983)

The relation of provable equivalence in Peano arithmetic is a universal ceer.

Γ-completeness

Let Γ be a complexity class (e.g., Σ^0_1, d-Σ^0_1, or Σ^0_n). We say that an equivalence relation R is Γ-**complete** (*for computable reducibility*) if:

- $R \in \Gamma$, and
- for any equivalence relation $E \in \Gamma$, we have $E \leq_c R$.

Note that the notions of a universal ceer and a Σ^0_1-complete equivalence relation are synonymous.

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Fact (folklore)

Let $X \subseteq \mathbb{N}$ be an oracle. Then there exists a $\Sigma^0_1(X)$-complete equivalence relation.

Proposition (Ianovski, R. Miller, Ng, and Nies 2014)

There exists a Π^0_1-complete equivalence relation. For any oracle X, there is no $\Pi^0_2(X)$-complete equivalence relation.

Let $a \in \mathcal{O}$. Recall that a set $A \subseteq \mathbb{N}$ is Σ^{-1}_a if there exist total computable functions $f(x,s)$ and $g(x,s)$ such that for all x and s,

(a) $f(x,0) = 0$ and $\lim_t f(x,t) = A(x)$;

(b) $g(x,0) = a$ and $g(x,s+1) \leq_{\mathcal{O}} g(x,s)$;

(c) if $f(x,s+1) \neq f(x,s)$, then $g(x,s+1) \neq g(x,s)$.

A set B is Π^{-1}_a if its complement $(\mathbb{N} \setminus B)$ is Σ^{-1}_a.

Proposition (Ng and Yu; B. and Kalmurzaev)

Suppose that $a \in \mathcal{O}$ and $|a|_{\mathcal{O}} \geq 1$. Then there are a Σ^{-1}_a-complete and a Π^{-1}_a-complete equivalence relations.

Problem

Find natural examples of Γ-complete equivalence relations.

Γ-complete equivalence relations in recursion theory

Recall that for a natural number e, $W_e = \{x \in \mathbb{N} : \varphi_e(x)\downarrow\}$.

Thus, one can treat, say, m-equivalence on c.e. sets as a relation on their indices: $i \sim_m j \ \Leftrightarrow\ W_i \equiv_m W_j$.

Γ-complete equivalence relations:

(1) 1-equivalence and m-equivalence on c.e. sets are both Σ^0_3-complete.

(2) Turing equivalence on c.e. sets is Σ^0_4-complete.

(1) Fokina, S.-D. Friedman, and Nies 2012

(2) Ianovski, R. Miller, Ng, and Nies 2014

Γ-complete equivalence relations in recursion theory

(3) For every $n \in \mathbb{N}$, 1-equivalence on (indices of) $\emptyset^{(n+1)}$-c.e. sets is Σ^0_{n+4}-complete.

(4) Fix an effective listing $(A_e)_{e\in\mathbb{N}}$ of all quadratic time computable languages $A \subseteq \{0,1\}^*$. Then the relation of almost equality

$$\{(i,j) : A_i =^* A_j\}$$

is Σ^0_2-complete.

(3)–(4) Ianovski, R. Miller, Ng, and Nies 2014

Computable structures and their indices

Recall that a number $e \in \mathbb{N}$ is a *computable index* of a computable L-structure $\mathcal{S}$ if the characteristic function of the atomic diagram $D(\mathcal{S})$ is equal to the partial computable function φ_e.

Thus, we can *identify* computable structures and their indices.

The isomorphism relation

Theorem (Fokina, S.-D. Friedman, Harizanov, Knight, McCoy, and Montalbán 2012)

For each of the following classes K of computable structures, the isomorphism relation on K is a Σ^1_1-complete equivalence relation:

- trees,
- graphs,
- torsion-free abelian groups,
- abelian p-groups,
- fields of any given characteristic,
- linear orders.

Hyperarithmetic isomorphism

Theorem (Greenberg and Turetsky)

The hyperarithmetic isomorphism relation among computable structures is a Π^1_1-complete equivalence relation.

Computable isomorphism

Theorem (Fokina, S.-D. Friedman, and Nies 2012)

For each of the following classes K of computable structures, the computable isomorphism relation on K is a Σ^0_3-complete equivalence relation:

- trees,
- equivalence structures,
- Boolean algebras.

Δ^0_α isomorphism

Theorem 1

Let α be a computable successor ordinal. For each of the following classes K of computable structures, the relation of Δ^0_α isomorphism is a $\Sigma^0_{\alpha+2}$-complete equivalence relation:

- distributive lattices,
- Heyting algebras,
- undirected graphs.

Proof sketch for $\alpha = 2$

(i) Any Σ^0_2 set $A \subseteq \mathbb{N}$ can be encoded via a computable sequence of linear orders

$$\mathcal{L}_n \cong \begin{cases} \omega + 1, & \text{if } n \notin A, \\ \omega + 2, & \text{if } n \in A. \end{cases}$$

This encoding is uniform in indices of Σ^0_2 sets.

(ii) Given an arbitrary Σ^0_2 equivalence relation E, one can produce a computable function $g(x)$ with the following property: $(x \, E \, y)$ iff there is a $\emptyset'$-computable permutation σ of $\mathbb{N}$ such that $\sigma(W^{\emptyset'}_{g(x)}) = W^{\emptyset'}_{g(y)}$.

(iii) After that, a set $W^{\emptyset'}_{g(x)}$, where $x \in \mathbb{N}$, is encoded via the sequence of linear orders from above. A computable distributive lattice $\mathcal{C}_x$ is defined as the direct sum of this sequence.

One can show that $(x \, E \, y)$ iff $\mathcal{C}_x$ and $\mathcal{C}_y$ are Δ^0_2 isomorphic.

Arithmetic isomorphism

Theorem 2

The relation of arithmetic isomorphism among computable partial orders is a Σ^0_ω-complete equivalence relation.

We conjecture that one can generalize the result as follows: Let λ be a computable limit ordinal. Then

$$\left(\text{being isomorphic via a function } f \in \bigcup_{\alpha<\lambda} \Delta^0_\alpha\right) \text{ is } \Sigma^0_\lambda\text{-complete.}$$

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