A unifying approach to Goodstein sequences

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Reuben L. Goodstein (1912-1985)

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Remark: For polynomials and exponential polynominals the k-normal forms also produce shortest possible terms representations for numbers.

Relating Goodstein to the Skolem class

Let Sk be the least set of unary functions such that $x \mapsto 0 \in \mathrm{Sk}$ and such that with $f, g \in \mathrm{Sk}$ we have
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It is open whether the full Skolem class induces a natural Goodstein principle.

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Proof: This is not straight forward. Monotonicity under base change fails e.g. for $m = A_1(k, A_0(k, A_1(k,\cdot)^{k-1}(1)))) > m-1$.

So we switch to new normal forms based on maximality under base change, prove termination for those Goodstein sequences and show that these dominate the ones under consideration.

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Proposition

$m < n \Rightarrow m[k \leftarrow k+1] < n[k \leftarrow k+1])$.

Define $m_0^{\max} := 0$ and for $m_l^{\max} > 0$
$m_{l+1}^{\max} := m_l^{\max}[l+2 \leftarrow l+3] - 1$ and 0 otherwise.

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$(\forall m)(\exists l)[m_l^{\max} = 0]$ (but this is unprovable in PA.)

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We firmly believe that the result we presented will lead to new notations system on natural numbers with intriguing properties.

Thank you for listening.