Plato and the foundations of mathematics

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On the uncountable

This talk reports on a spin-off of a joint project with Dag Normann (U. of Oslo) on the Reverse Mathematics and computability theory of the uncountable.

On the uncountable

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See arXiv for some of our papers!

Plato and his -ism

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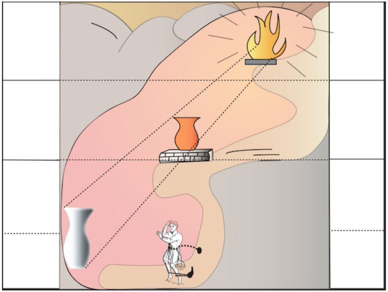
Plato's allegory of the cave provides a powerful visual:



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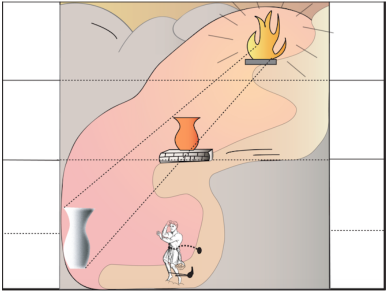

We can only know reflections/shadows/... of ideal objects.

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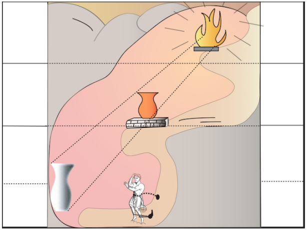

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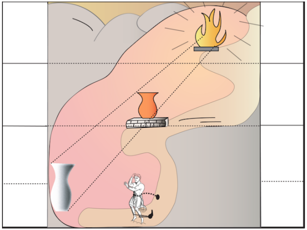

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What are the current foundations of mathematics reflections of?

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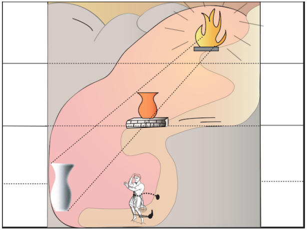

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What are the current foundations of mathematics reflections of?

X?

Gödel hierarchy

Reverse Mathematics, etc

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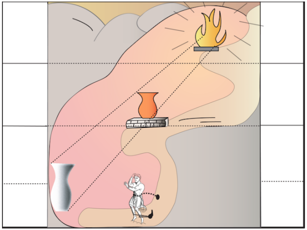

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What are the current foundations of mathematics reflections of?

X?

↙

Gödel hierarchy
Reverse Mathematics, etc

This talk: find X and the associated embedding '↙'.

Gödel hierarchy

strong:
- ⋮
- large cardinals
- ⋮
- ZFC
- ZC (Zermelo set theory)
- simple type theory

medium:
- Z_2 (second-order arithmetic)
- ⋮
- Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
- Π^1_1-CA_0 (comprehension for Π^1_1-formulas)
- ATR_0 (arithmetical transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak:
- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

It is striking that a great many foundational theories are linearly ordered by [consistency strength] $<$. Of course it is possible to construct pairs of artificial theories which are incomparable under $<$. However, this is not the case for the "natural" or non-artificial theories which are usually regarded as significant in the foundations of mathematics.
(Simpson, Gödel Centennial Volume; also: Koellner, Burgess, Friedman,. . .)

Gödel hierarchy

= 'comprehension' hierarchy

↑

MORE sets exist

...

LESS sets exist

↓

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Gödel hierarchy

Zermelo-Fraenkel set theory with choice
aka 'the' foundation of mathematics

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- Z_2 (second-order arithmetic) — Hilbert-Bernays’s Grundlagen der Mathematik
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Hilbert-Bernays's Grundlagen der Mathematik

Russell-Weyl-Feferman predicative mathematics

weak

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The 'Big Five' of Reverse Mathematics

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	strong	⋮ large cardinals ⋮ ZFC
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The 'Big Five' of Reverse Mathematics		
Hilbert's finitary math	weak	WKL_0 (weak König's lemma) RCA_0 (recursive comprehension) PRA (primitive recursive arithmetic) bounded arithmetic

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Received view: natural/important systems form linear Gödel hierarchy

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Received view: natural/important systems form linear Gödel hierarchy and 80/90% of ordinary mathematics is provable in ACA_0/Π^1_1-CA_0.

Today: a higher RM

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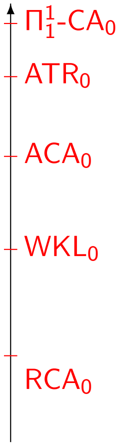

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Π^1_1-CA_0

ATR_0

ACA_0

WKL_0

RCA_0 proves Δ^0_1-comprehension

Today: a higher RM

$\Pi_1^1\text{-CA}_0$

ATR_0

ACA_0

WKL_0 $\leftrightarrow$ Dini's theorem.

$\leftrightarrow$ countable Heine-Borel compactness

$\leftrightarrow$ Riemann int. thms

RCA_0 proves Δ_1^0-comprehension

Today: a higher RM

- Π_1^1-CA_0
- ATR_0
  - $\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
- ACA_0 $\leftrightarrow$ Monotone conv. thm
  - $\leftrightarrow$ Ascoli-Arzela
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SECOND-ORDER arithmetic

BOOT_2

Σ-TR

BOOT

HBU

RCA_0^ω

HIGHER-ORDER arithmetic

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HIGHER-ORDER arithmetic

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WKL_0	$\leftrightarrow$ Dini's theorem.	HBU	$\leftrightarrow$ Dini's theorem for nets.
	$\leftrightarrow$ countable Heine-Borel compactness		$\leftrightarrow$ uncountable Heine-Borel compactness
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SECOND-ORDER arithmetic

$BOOT_2$

Σ-TR

$\leftrightarrow$ range of $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ exist

BOOT $\leftrightarrow$ Mon. conv. thm for nets

$\leftrightarrow$ Ascoli-Arzela for nets

HBU $\leftrightarrow$ Dini's theorem for nets.

$\leftrightarrow$ uncountable Heine-Borel compactness

$\leftrightarrow$ gauge integral thms

RCA^ω_0 proves Δ-comprehension

HIGHER-ORDER arithmetic

Today: a higher RM

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ATR_0	$\leftrightarrow \dots$	Σ-TR	$\leftrightarrow \dots$
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Today: a higher RM

ECF replaces uncountable objects by countable representations/RM-codes

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HIGHER-ORDER arithmetic

ECF $\leftarrow$

Today: a higher RM

ECF replaces uncountable objects by countable representations/RM-codes

ECF converts right-hand side to left-hand side, including equivalences!

SECOND-ORDER arithmetic		HIGHER-ORDER arithmetic
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RCA_0 proves Δ^0_1-comprehension	ECF ←	RCA_0^ω proves Δ-comprehension

Cousin's lemma and HBU

Ordinary mathematics = prior to or independent of abstract set theory

Cousin's lemma and HBU

Ordinary mathematics = prior to or independent of abstract set theory

Weierstrass (1880) and Pincherle (1882) used compactness, but did not explicitly formulate it. Cousin (1893) proves Cousin's lemma.

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$$(\forall \Psi : I \to \mathbb{R}^+)(\exists y_1, \dots, y_k \in [0,1])([0,1] \subset \cup_{i \leq k} I_{y_i}^\Psi) \qquad \text{(HBU)}$$

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The reals $y_1, \dots, y_k$ yield a finite sub-cover; NO conditions on Ψ.

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PS: Borel's proof of HBU ($\approx$ 1900) makes no use of the axiom of choice. With minimal adaption, Borel's proof yields a realiser for HBU.

Beyond Riemann and Lebesgue: the gauge integral

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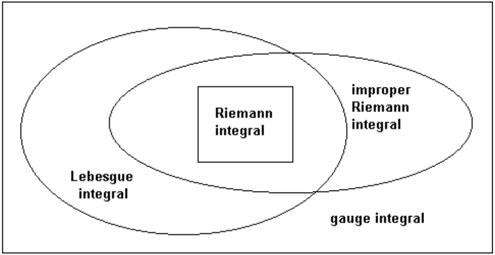

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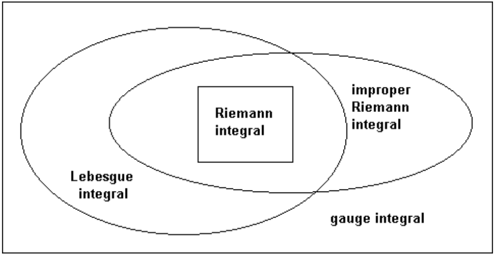

The first step in gauge integration is always Cousin's lemma!

The gauge integral: Riemann's cousin!

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Definition (Riemann integral)

$f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable on $I \equiv [0,1]$ with integral $A \in \mathbb{R}$:

$$(\forall \varepsilon > 0)(\exists \underbrace{\delta > 0}_{\textit{constant}})(\forall P)(\underbrace{(\forall i \leq k)(|x_i - x_{i+1}| < \delta)}_{P \text{ is 'finer' than } \delta} \to |S(P,f) - A| < \varepsilon)$$

$P = (0, t_1, x_1 \ldots x_k, t_k, 1)$ partition $S(P,f) = \sum_i f(t_i)|x_{i+1} - x_i|$ Riemann sum

The gauge integral: Riemann's cousin!

Definition (Riemann integral)

$f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable on $I \equiv [0,1]$ with integral $A \in \mathbb{R}$:

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A function is f Lebesgue integrable IFF f and $|f|$ are gauge integrable.

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A strong version of FTC is equivalent to HBU over RCA_0^ω.

Today: a higher RM

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HIGHER-ORDER arithmetic

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A set $D \neq \emptyset$ with a binary relation '$\preceq$' is directed if

- a. The relation $\preceq$ is transitive and reflexive.
- b. For $d, e \in D$, there is $f \in D$ such that $d \preceq f \wedge e \preceq f$.

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Sequences are nets for $(D, \preceq) = (\mathbb{N}, \leq)$. We write x_d for $x(d)$.

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Adding a modulus of convergence to $\mathsf{MCT}_{\mathsf{net}}$ yields equivalence to BOOT + $\mathsf{QF}\text{-}\mathsf{AC}^{0,1}$. Non-uniqueness and ECF!

The Gödel and Plato hierarchies

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SECOND-ORDER arithmetic		ECF ←	THIRD-ORDER arithmetic	
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HBU, BOOT, etc are provable in Hilbert-Bernays' Grundlagen der Mathematik

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Specker nets, fields and rings over $\mathbb{N}^{\mathbb{N}}$, ...

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The previous also works for all finite types. E.g. monotone convergence for nets indexed by $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, is at the level of Π^1_2-CA_0.

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Any (content) questions?