Some questions of uniformity in algorithmic randomness

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The central theorem

Let α be a lower-semicomputable (or left-c.e.) real in $[0,1]$. The following are equivalent:

(i) α is Martin-Löf random

(ii) $\alpha = \Omega_U = \sum_n 2^{-K_U(n)}$ for some universal prefix-free machine U

(iii) $\alpha = \sum_n m(n)$ for some maximal lower-semicomputable semi-measure m

(iv) α is Solovay-complete (maximal for the Solovay order, where $\beta \leq_S \alpha$ if $k\alpha - \beta$ is left-c.e. for some integer k)

(Combination of several results by Chaitin, Solovay, Calude et al., Kučera and Slaman).

The Kraft-Chaitin theorem

On the other hand, the Kraft-Chaitin theorem tells us that, if we are not concerned with universality, we have the **uniform** equivalence:

(i) α is left-c.e. in $[0, 1]$

(ii) $\alpha = \Omega_M = \sum_n 2^{-K_M(n)}$ for some prefix-free machine M

(iii) $\alpha = \sum_n m(n)$ for some lower-semicomputable semi-measure m

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- A universal prefix-free machine U such that $\sum_n 2^{-K_U(n)} = \alpha$?

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- A universal prefix-free machine U such that $\sum_n 2^{-K_U(n)} = \alpha$?
- A maximal lower-semicomputable semimeasure m such that $\sum_n m(n) = \alpha$?

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We build a left-c.e. random α. By the recursion theorem we can know its index and thus know which machine *U* we are up against.

We build our own prefix-free machine *M* and want to ensure that
either $\sum_n 2^{-K_U(n)} \neq \alpha$
or there is no constant *c* such that $K_U \leq K_M + c$.

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Thus we can assume that the opponent "stays below us" ($\Omega_U < \alpha$) at all times.

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Let us fix a c and try to deal with one requirement.

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As we move towards β, we monitor Ω_U which as discussed must remain below us. We wait for a stage where Ω_U gets very close to α (say, 2^{-c-k-4}-close).

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This puts the opponent in a bad spot. Either he tries to match our description by issuing a new $U(p) = \sigma$, with $|p| \leq |0^k 1| + c$, but then Ω_U becomes greater than α! (and we win). Or he never matches our description and we satisfy our requirement by ensuring $K_U(\sigma) > K_M(\sigma) + c$

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Then either the opponent follows us, and we pick a new intermediate β' while having made progress towards the real β. Or he does not and the requirement is satisfied.

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Example: $F(x) = \sum_n \frac{(-1)^{x(n)}}{n}$.

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Theorem

Yes and no.

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And the following stronger theorem holds:

Theorem

One cannot $\emptyset$'-uniformly produce an index of a universal prefix-free machine U from the index of a Martin-Löf random α.

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We start our construction in the interval $[\xi_0, \xi_1]$. If at any point there is a mind change on the index of the machine U, we move to $[\xi_1, \xi_2]$ and restart the construction against the new machine.

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We start our construction in the interval $[\xi_0, \xi_1]$. If at any point there is a mind change on the index of the machine *U*, we move to $[\xi_1, \xi_2]$ and restart the construction against the new machine.

If there are finitely many mind changes, we get to perform our construction. And if there are infinitely many mind changes, our final α will be ξ while the opponent will have failed to $\emptyset'$-produce the index of a machine *U*.

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This follows from previous work. Let Ω be the halting probability of an optimal machine. Kučera and Slaman showed how from the index of a left-c.e. real $\alpha \in [0, 1]$ one can build a Martin-Löf test $(\mathcal{V}_k)$ such that if $\alpha \notin (\mathcal{V}_k)$ then one can, uniformly in *k*, produce approximations $\alpha_1 < \alpha_2 < \ldots$ of α and $\Omega_1 < \Omega_2 < \ldots$ of Ω such that $(\alpha_{s+1} - \alpha_s) > 2^{-k}(\Omega_{s+1} - \Omega_s)$.

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Then, as shown by Calude et al., one can use such approximations to uniformly build a uniform machine with halting probability α.

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- There exists a machine V such that $\Omega_U - \Omega_V$ is neither left-c.e. nor right-c.e.
- If U is universal, then V can be taken to be universal as well.

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Question (Barmpalias and Lewis-Pye): how uniform is this theorem?

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Theorem

The construction of V from U is not uniform in general, but ***is*** *uniform if U is universal.*

Děkuji

thank you